Complexity and Prediction Part V: The 'foundational crisis of mathematics', set theory, Hilbert, Gödel, and Turing
Version 1, May 2015*

Abstract

This paper sketches some history of set theory, the crisis over the logical foundations of mathematics at the end of the 19th century, 'the Hilbert programme', the *Entscheidungsproblem* (decision problem), and the famous papers of Gödel (1931) and Turing (1936) demonstrating some fundamental properties of logic, mathematics, and computation. Apart from its inherent fascination, this story is interesting background to the development of modern digital computers in the 1940s and to the inter-disciplinary study of complex systems connecting mathematics, logic, physics, biology, cognitive science, economics, computer science, and machine intelligence that developed after 1945. A fundamental issue in science and politics is *prediction*. This story helps understanding of contemporary debates about possibilities for new tools to help successful prediction.

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'What has been said once can always be repeated.' Zeno.

'We must endeavour that those who are to be the principal men of our state go and learn arithmetic, not as amateurs, but they must carry on the study until they see the nature of numbers with the mind only… [A]rithmetic has a very great and elevating effect.' Plato.

'I am so in favour of the actual infinite that instead of admitting that Nature abhors it, I hold that Nature makes frequent use of it everywhere.' Leibniz.

'No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed, as long as he is not led by an acquired habit of mind to regard the infinite as something bounded.' Gauss.

'The very possibility of a science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigour which no one dares to doubt? If, on the contrary, all the propositions it enunciates can be deduced one from the other by the rules of formal logic, why is not mathematics reduced to an immense tautology? … Shall we then admit that the theorems which fill so many volumes are nothing but devious ways of saying that A is A?' Poincaré, 1896.

[A] complete epistemological description of a language A cannot be given in the same language A, because the concept of truth of sentences of A cannot be defined in A. It is this theorem that is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic.' Kurt Gödel, explaining Turing's 1936 paper.

'Gödel's achievement in modern logic is singular and monumental – indeed it is more than a monument, it is a land mark which will remain visible far in space and time. Gödel's results showed, to the great surprise of our entire generation, that [Hilbert's Programme] could not be implemented.' Von Neumann.

'Gödel's theorem is an inexhaustible source of intellectual abuses.' Alan Sokal.

'The world is rational' (Die Welt ist vernunftig), Gödel.

'To view the Church-Turing hypothesis as a physical principle does not merely make computer science a branch of physics. It also makes part of experimental physics into a branch of computer science.' Deutsch.

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This series of blogs and notes explores the themes of *complexity and prediction*.

Part I: Introduction.

Part II: Controlled skids and immune systems. Why is the world so hard to predict? Nonlinearity and Bismarck. How do humans adapt? The difference between science and political predictions. Feedback and emergent properties. Decentralised problem-solving in the immune system and ant colonies. Some problems with political decision-making and institutions.

Part III von Neumann and economics as a science. This examines von Neumann's views on the proper role of mathematics in economics and some history of game theory.

Part IV: Leibniz and computational thinking. The first computers. Punched cards. Optical data networks since Homer. Wireless. The state of the art by the time of Turing's 1936 paper.

This is part of a project with two main aims. They are, first, to sketch *a new approach to education and training* and, second, to suggest a new approach to political priorities in which *progress with education and science becomes a central focus for the British state*. The two are entangled: progress with each will hopefully encourage progress with the other.

We need experiments with what the Nobel-winning physicist Murray Gell Mann (namer of the 'quark') called an 'Odyssean' education synthesising a) maths and the natural sciences, b) the social sciences, and c) the humanities and arts, into crude, trans-disciplinary, integrative thinking. This should combine courses like Berkeley's '*Physics for Future Presidents*' and Fields Medallist Timothy Gowers' equivalent for maths (cf. the new MEI/OCR course based on his blog) with the best of the humanities; add new skills such as coding; and give training in managing complex projects and using modern tools (e.g agent-based models). Universities should develop alternatives to *Politics, Philosophy, and Economics* such as *A History of Ideas, Physics for Future Presidents, and Running a Start-up*. We need leaders with an understanding of Thucydides and statistical modelling, who have read *The Brothers Karamazov* and *The Quark and the Jaguar*, who can feel Kipling's *Kim* and succeed in Tetlock's *Good Judgement Project*. An Odyssean education would focus on humanity's most important problems and explain connections between them to train *synthesisers*. I explain what Gell Mann meant about an 'Odyssean' approach HERE (p. 5-8). In future blogs I will sketch some ideas for practical training courses.

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The study of *complex systems* cuts across many fields including mathematics, logic, physics, biology, cognitive science, economics, computer science, and machine intelligence. It is necessary to understand something of these cross-disciplinary developments in order to understand many contemporary developments in science, technology, economics, and politics and new tools that are being developed that will affect society and politics profoundly. Many aspects of contemporary 'political philosophy' (much of which has not incorporated scientific knowledge), and many contemporary political issues, implicitly rely on ideas concerning *what it is possible to predict* so it is useful, as well as interesting, to consider some foundations of our knowledge.

The story of how computers and computer science emerged from the most esoteric of subjects, mathematical logic, is not only inherently fascinating but explains some of the intellectual background to 20th Century thinking about *complexity and prediction*. At the start of the 20th Century, various schools of thought were fighting over the 'foundational crisis of mathematics' —

the problem of how to set mathematics on completely secure logical foundations. There had been 'crises' in maths before. For example, the Pythagoreans' discovery of the irrational numbers was believed to have sparked murder. The discovery by Newton and Leibniz of calculus and infinitesimals (which many, such as Bishop Berkeley, said was nonsense) provoked a crisis. In the 19th Century, the discovery of non-Euclidean geometries by Gauss and others provoked another crisis. By the end of 1903, the entire logical structure of mathematics was felt to be tottering because of problems with *infinities* and *set theory*.

Over the next thirty years, there were many attempts to find a completely secure logical foundation for mathematics and banish the paradoxes of set theory. In the 1930's came two astonishing papers that resolved many of the problems and will be remembered long after almost everything else in the century is forgotten: Kurt Gödel's *On Formally Undecidable Propositions of* Principia Mathematica *and Related Systems I* (1931) and Alan Turing's *On Computable Numbers, With an Application to the Entscheidungsproblem* (1936). These papers answered definitively certain aspects of questions that have been asked since the pre-Socratics such as: *can mathematics provide absolutely reliable knowledge or is 'man the measure of all things' as Protagoras claimed?*

They also laid the foundations for theoretical and practical work by Turing and John von Neumann in the 1940s and 1950s on machine intelligence and building computers as they hopped back and forth between the esoteric world of mathematical logic to the business of defeating Nazi Germany. This work sparked and merged with other inter-disciplinary ideas on how to simulate and thus predict the dynamics of complex systems such as the brain, the economy, weapon systems, and the weather. When one reads today about cutting-edge developments in 'agent-based simulations' to model financial markets or epidemics, many of the ideas flow directly from these intellectual foundations built 1930-1955. (Those on funding committees take note how important it is not to follow fashion, or what Feynman called 'the comet head'.)

In a later blog in this series, I will sketch the next chapters in this story. It has turned out that the fields of *mathematical logic, computation,* and *quantum mechanics* are intimately related — a wondrous example of what Wigner famously called 'the unreasonable effectiveness of mathematics'.

In 1956, von Neumann was lying on his deathbed, dying tragically young, when he received a letter from Kurt Gödel that first stated what is now known as the P=NP? problem, along with the Riemann Hypothesis one of the most important problems in mathematics. This raised the possibility that algorithms might exist that can *solve* problems as quickly as they can be *verified* which would have immense practical, as well as theoretical, implications. Imagine *solving* the Riemann Hypothesis as quickly as a proof can be *verified*. From logistics to cryptography the world would be transformed if P=NP.

In 1981 Feynman published his classic paper, *Simulating Physics with Computers*, on the possibility of quantum computation and in1985 David Deutsch extended Turing's 1936 proof concerning classical computation to the quantum realm: a quantum computer can simulate any physical process. Seth Lloyd, another pioneer in this field, sums it up by saying that one can think of the universe as a quantum computer. In the 1990s it was proved that quantum computers could solve certain classes of mathematical problems much faster than classical computers. One of the most important theoretical and practical questions is: can we build large-scale quantum computers and what sort of problems, with differing degrees of computational complexity, might they solve?

I wrote some of these notes for myself as I studied some of these topics. I cannot understand the development of ideas without understanding the chronological story but usually the books on such topics are technical or non-chronological or both. These notes are an attempted summary of this story. Having written them, and having discussed such things with people in education, I thought they may be useful particularly for a subset of 15-25 year-olds who want to understand the story chronologically. As a preface to this blog, it may be useful to skim through [this preceding one](), particularly the sketches on Leibniz and Boole.

* NB. These notes were written before May 2015 but I have put them on this blog without having finished them properly because of engagement with another project. I will update this version with corrections when I have the time. Please leave corrections in Comments on my blog.

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The sections in this note are:

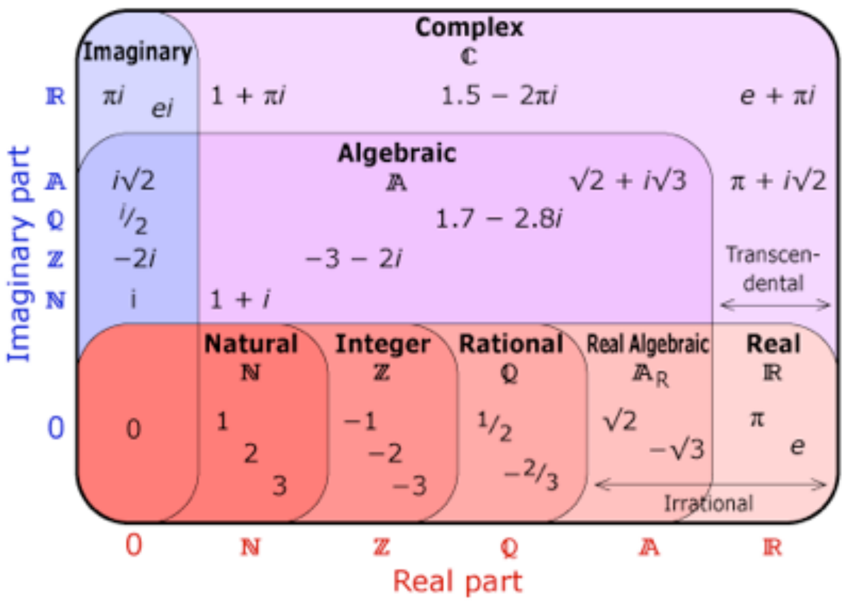
A lot of this story could be taught in schools yet almost none of it is. The current exam system sadly deems that only those doing Further Maths A Level (~2% of the cohort) are taught about complex numbers. Other topics are similarly restricted. For example, many schools, until the disastrous influence of the National Curriculum and GCSEs, taught children under 13 about *matrices* but these are now also reserved for Further Maths A Level (outside very unusual schools). The *normal distribution* is reserved for the S2 paper in A Level Mathematics.

There is no educational justification for this. It is partly a product of the control of the exam system by Whitehall and MPs and the fact that among those with power in the education system the overriding goal of the vast majority is to keep everybody doing the *same* government controlled exams at 16. The goal is *not* to maximise learning for all points on the IQ distribution. Those with power want exams that are *politically and bureaucratically optimal* according to their priorities, *not* exams that are *educationally optimal*. Good schools should ignore GCSEs and teach very different curricula. For example, Marcus du Sautoy's *The Story of Mathematics* Open University course is suitable for many school children.

In the Department for Education 2010-14, Michael Gove's team tried to end GCSEs and start a process for very different exams. We managed to make some worthwhile reforms of GCSEs and the curriculum, such as making computer science a mainstream subject, but we only did a small part of what is needed. We need people outside Westminster and Whitehall to develop new exams including the equivalent of the STEP exam in Maths for other subjects at age 16.

Some basic concepts concerning number and sets

Picture: A Venn diagram of the number system - real, complex, algebraic, transcendental



A *set* is '*denumerable*' (or 'countable') if it can be put into one-to-one correspondence with the positive integers. It is '*non-denumerable*' (or 'uncountable') if it is infinite and cannot be put into one-to-one correspondence with the positive integers. All finite sets are 'denumerable'. Some infinite sets are defined as 'denumerable' and others as 'non-denumerable'. The integer, rational and 'algebraic' numbers are denumerable; the irrational, 'transcendental', real, and complex numbers are non-denumerable.[1] The terminology here can be very confusing. The fact that a set is defined as 'countable' does *not* necessarily mean that you could count it in the normal sense of the word, even if you lived for ever, nor does it mean that in principle there could be a physical list, since for infinite sets there is not enough space in the universe. The importance of this distinction between denumerable/non-denumerable is explained below.

An *integer* is a positive or negative whole number. A *natural* number is a positive whole number. A *rational* number is expressible as a ratio of integers such as m/n when m and n are natural numbers. As decimals,[2] they either terminate (e.g ½ = 0.5) or repeat a pattern infinitely (e.g. 1/11 = 0.0909…); the decimal expression of any rational number is *periodic*. Rational numbers could be incorporated in the five basic laws of arithmetic.[3] The acceptance of zero and negative numbers

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[1] For a short formal explanation of some of these concepts see [this blog](#) by Tim Gowers.

[2] The choice of decimals as a 'base' does not affect the fundamental definitions or arguments concerning types of number or the countability of sets. In his classic book '*Number*', Dantzig discusses the physiological accident (10 fingers) that we now use base 10. If left to engineers, he argues, we would probably use a base with more divisors (such as 12); if left to mathematicians, we would probably use a base with a prime number such as 7 or 11, because with a prime base 'every systematic fraction would be irreducible' and we would not have fractions such as 0.36 = 36/100 = 18/50 = 9/25. 'So may the decimal system stand as a living monument to the proposition: "Man is the measure of all things."'

[3] The five basic laws of arithmetic are: a+b=b+a; a+(b+c)=(a+b)+c; ab=ba; a(bc)=(ab)c; a(b+c)=ab+ac.

required new rules of arithmetic to maintain the consistency of the old.[4] Generalising from the natural to the rational numbers 'satisfies both the theoretical need for removing the restrictions on subtraction and division, and the practical need for numbers to express the results of measurement. It is the fact that the rational numbers fill this two-fold need that gives them their true significance… The inherent human tendency to cling to the "concrete", as exemplified by the natural numbers, was responsible for this slowness in taking an inevitable step [ie. extending to rational numbers]. Only in the realm of the abstract can a satisfactory system of arithmetic be created'.[5]

The rational points are 'dense on the number line'; in any interval, however small, we can always find another rational point because we can simply increase the size of the denominator, so 'there is no interval on the line ... free from rational points' and 'there must be infinitely many points in any interval' (Courant). From a practical point of view the rational numbers are all we need for measuring things in the physical world.[6] *The set of rationals is countable.*

An *irrational* number is not rational; it is a number that cannot be expressed as *m*/*n* where *m* and *n* are integers.[7] They are infinite decimals that do not repeat a pattern. √2 is irrational as the Pythagoreans first showed (and is 'algebraic'); irrational square roots (√) were named 'surds'.[8] Since the set of rational points covers the line densely, it seems that all points must be rational points but the discovery of irrational numbers showed that this reasonable conclusion is wrong: although the rational points are everywhere dense, they do not cover all of the number line. 'Nothing in our "intuition" can help us to "see" the irrational points as distinct from the rational ones. No wonder that the discovery of the incommensurable [irrational] stirred the Greek philosophers and mathematicians…' (Courant).[9] *The set of irrationals is non-denumerable.*

An *algebraic* number. If *r* is a root of a nonzero polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

where the a_is are integers (or equivalently, rational numbers) and *r* satisfies no similar equation of degree $< n$, then *r* is said to be an algebraic number of degree n (Wolfram definition).

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[4] The acceptance of zero and negative numbers required rules such as (-1)(-1)=+1, and we forbid the operation x/0 in all circumstance, to make them compatible with the 'rational operations' and the five basic laws (we could not maintain a(b+c)=ab+ac and allow (-1)(-1)=-1, and allowing division by 0 would mean we could deduce from the true equation 0x1=0x2=0 the absurdity 1=2). Rules for negative numbers cannot be 'proved'; they are 'created by us in order to attain freedom of operation while preserving the fundamental laws of arithmetic' (Courant).

[5] Courant, p.56.

[6] 'Since the rational numbers are dense on the line, it is impossible to determine by any physical operation, however precise, whether a given length is rational or irrational. Thus it might seem that the irrational numbers are unnecessary for the adequate description of physical phenomena. But … the real advantage which the introduction of irrational numbers brings to the mathematical description of physical phenomena is that this description is enormously simplified by the free use of the limit concept, for which the number continuum is the basis.' Courant

[7] 'An irrational point is completely described by a sequence of nested rational intervals with lengths tending to zero.' Courant, p. 69.

[8] According to legend the Pythagorean scholar Hippasus discovered that √2 is an *irrational number* and that there are geometrical lengths that cannot be expressed as the ratio of two whole numbers. Such was the crisis created by the discovery of irrational numbers (or 'incommensurable' numbers as we sometimes call them now; the Greeks called them *algon*, or 'unspeakable') that he was supposedly drowned by the Pythagoreans.

[9] 'From a purely formal point of view, we may start with a line made up only of rational points and then *define* an irrational point as just a symbol for a certain sequence of nested rational intervals. An irrational point is completely described by a sequence of nested rational intervals with lengths tending to zero.' Courant, p.69.

All rational numbers are algebraic; some irrationals are algebraic. Algebraic numbers can be *real* (like $\sqrt{2}$ (since it is the root of x^2-2) or *complex* (like *i*). *The set of algebraic numbers is denumerable.*

A *transcendental* number is a non-algebraic number. It is a number that cannot be produced by (is not 'the root' of) any polynomial equation with integer coefficients, hence it 'transcends' algebra. Most reals are transcendental; some transcendental numbers are real and some are complex. Given a rational number is algebraic by definition, every real transcendental number must be irrational. *The set of transcendental numbers is non-denumerable.*

Although it is usually hard to show that a specific number is transcendental (and we only have a few concrete examples) transcendentals nevertheless make up almost all real and complex numbers.

> 'The algebraic numbers are spotted over the plane like stars against a black sky; the dense blackness is the firmament of the transcendentals.' (Bell)

The name 'transcendental' comes from Leibniz. In 1737 Euler proved *e* is irrational. In 1768, Lambert proved that π is irrational. Euler conjectured that there are points on the number line that are *not algebraic* but he could not find any (although he knew about e he did not know it was transcendental). Liouville (who helped salvage Galois's work after his death in a duel) proved the existence of transcendental numbers in 1844, proved that there are infinitely many of them, and gave the first example of one in 1851.[10] In 1873, Hermite proved that *e* is transcendental. In 1882, Lindemann (Hilbert's supervisor) proved that π is transcendental, 'thus did modern analysis dispose of a problem which had taxed the ability of mathematicians since the days of Thales' and this ended 'the second attempt to exhaust nature by number' (Dantzig).[11]

> 'The discovery of transcendentals, the establishment of the fact that they are far richer in extent and variety than the irrationals of algebra, that they comprise some of the most fundamental magnitudes of modern mathematics - all of this showed definitely that the powerful machinery of algebra had failed just where the elementary tools of rational arithmetic had failed two thousand years earlier. Both failures were due to the same source: algebra, like rational arithmetic, dealt with *finite processes*.
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> 'Now, as then, infinity was the rock which wrecked the hope to establish number on a firmer foundation. But to legalize infinite processes, to admit these weird irrational creatures on terms of equality with rational numbers, was just as abhorrent to the rigorists of the nineteenth century as it had been to those of classical Greece.
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> 'Loud among these rose the voice of Leopold Kronecker, the father of modern intuitionism. He rightly traced the trouble to the introduction of irrationals and proposed that they be banished from mathematics. Proclaiming the absolute nature of the integers he maintained that the natural domain, and the rational domain immediately reducible to it, were the only solid foundation on which mathematics could rest. "God made the integer, the rest is the work of man," [said Kronecker]' Dantzig, p. 122.

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[10] One of the three famous ancient Greek problems was 'circle squaring'. For a number to be produced by Greek geometry rules it had to be either rational or a particular kind of algebraic. The proof of the existence of transcendentals in the 19th Century therefore also proved that the ancient Greek problem was insoluble.

[11] Many related questions like - *are* π+e *or* π/e *transcendental?* - remain unsolved.

Some of these issues are discussed in the next section.

A *real* number is a number that can be given to any desired degree of decimal places. Real numbers can be positive or negative, rational or irrational, algebraic (like $\sqrt{2}$) or transcendental (like e or π). The *real domain* has no first or last member: it stretches from $-\infty$ to $+\infty$. The real domain is 'well-ordered': one can always tell which is the greater. The rational domain is a sub-section of the real domain. All real numbers lie on the *real number line* - the x-axis in the Gauss-Argand diagram (below). The aggregate of real numbers is 'everywhere dense'; i.e. between any two real numbers an infinite number of other reals can be inserted. The real numbers are a subset of the complex numbers.

A *complex* number. Complex numbers were first described in the 16th Century by the criminal, Cardano, as he attempted to find a general formula for cubic equations. They have been very controversial but after centuries of being confined to disputes in pure mathematics, they would suddenly prove vital to the mathematical formulations of quantum mechanics in the 1920's.

> '[T]he Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity.' Leibniz

Cardano considered algebraic problems that seemed to yield *negative square roots*. The set of real numbers is, despite being uncountable, not sufficient to solve even very simple equations such as $x^2 + 1 = 0$ (i.e. $x^2 = -1$, but any real number squared gives either 0 or a positive number, so in school we are taught to answer 'no real solution'). To solve such equations, we need to consider something that in primary school we think of as impossible: $\sqrt{-1}$. Mathematicians did not even write $x=\sqrt{-1}$ since it seemed so meaningless until Cardano, just as they had previously shied away from using negative numbers. Cardano did not know what to do with such an odd creature and said of the idea of an imaginary number that it was 'as subtle as it is useless'. Bombelli developed the idea further (driven by work on Cardano's cubic formula) and said that the number $\sqrt{-1}$ was something that 'seemed to rest on sophistry rather than on truth'.[12]

The number designated by *i* is an '*imaginary*' number: $i=\sqrt{-1}$ and $i^2 = -1$.[13] A *complex* number, *z*, consists of a real and imaginary part: $z = a + bi$ where *a* and *b* are real numbers. (A real number is a complex number with an imaginary part of zero: e.g. 2 = 2+0i.) If one forgets for the moment what such expressions 'mean', one can try manipulating, say, $(2+3i) + (4+5i) = 6+8i$; or $(2+3i) \times (4+5i) = (8+10i+12i+15i^2) = (8+22i+15(-1)) = -7+22i$.

Dantzig wrote that when those who do not know the whole story look at maths and its textbooks, it seems that our understanding must have advanced in a certain logical sequence but this is wrong. In fact intuition played the dominant role and 'distant outposts were acquired before the intermediate territory had been explored'.

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[12] Devlin gives a slightly different account (*The Millennium Problems*, 2002). Cardano found that his method for solving cubics involved intermediate steps that involved the square root of negative numbers 'even though the final solutions to the cubic were real.' He decided that although one could not have a square root of a negative number as the final outcome of a calculation, it was permissible to use them in intermediate steps. 'This could happen if any occurrence of a square root of a negative number in an intermediate step was subsequently squared. For example, if $\sqrt{-3}$ arose in an intermediate step, squaring it would produce -3, which is real.' Cardano called such intermediate steps 'sophistic'. The term 'imaginary' was first used by Euler in 1770.

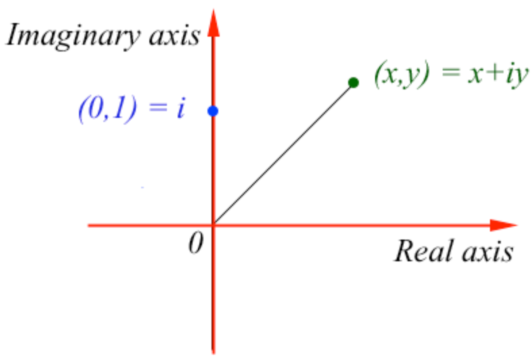
[13] So $i^3 = i \times i^2 = i \times (-1) = -i$. $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$. $i^5 = i \times i^4 = i$. $i^6 = i \times i^5 = i^2 = -1$. If you imagine these plotted on the Gauss-Argand diagram below, you will see that the powers of *i* represent points on a circle with origin 0,0, and multiplications of *i* can be thought of as *rotations*.

> 'It was the function of intuition to create new forms; it was the acknowledged right of logic to accept or reject these forms, *in whose birth it had no part*. But the decisions of the judge were slow in coming, and in the meantime the children had to live… The evolution of the complex number concept, this weirdest chapter in the history of mathematics, bears all the marks of such a development.'

We did not wait for Cantor and Dedekind to establish real numbers on a sure logical footing in the late 19th Century before exploring complex numbers.

In the 17th Century, Wallis realised that one could not place an imaginary number on the normal number line of real numbers because it is neither positive nor negative, so he added a new vertical number line for the imaginary numbers, intersecting the real number line at 0 (because *0* times *i* must equal 0). Complex numbers are part real and part imaginary so Wallis said that we should get the real part from the *x* axis and the imaginary part from the *y* axis; that is, we should consider complex numbers as distributed on a 2D plane, not ordered on a line like reals. He was ignored and the ideas were independently re-discovered and absorbed in the early 19th century. In his doctoral thesis (aged 20) on *the fundamental theorem of algebra*, Gauss 'implicitly used a geometrical representation of the complex domain' and in 1831 published a paper showing the equivalence of plane Cartesian geometry with the domain of the complex number (Dantzig p. 208). Cauchy, Weierstrass, Riemann, and others then extended the apparatus of infinite processes to complex numbers ('complex analysis') while others used complex numbers to develop a general *projective geometry* and Riemann ideas about non-Euclidean geometry that Einstein would grasp when struggling for the equations of General Relativity.

The Gauss-Argand Diagram for complex numbers, where the x-axis represents the real number line, the y-axis represents the imaginary number line, $i=\sqrt{-1}$*, and a complex number z is defined as* $z = x+iy$[14]



In 1748, Euler published his *Introductio ad analysin infinitorum* (*Introduction to Infinite Analysis*), which played a similar role for analysis as *Elements* had for geometry and *Al-Jabr* had for algebra (Crease). In it Euler reorganised 'analysis' which grew out of calculus and the study of irrational numbers, imaginary numbers, functions, and infinite series. In *Introductio,* Euler set out much of what is now covered in the introductory 'pure' modules of the modern Maths A Level such as trigonometric functions, infinite series, and exponential functions, and he revealed various properties of e and the 'natural logarithm' (ln = $\log_e$).

Euler also discovered an equation often described as 'the most beautiful equation in mathematics'.

[14] Wessell first thought of this sort of representation in 1799 but his article was ignored for a century and Argand independently rediscovered it in 1810. Gauss first used the term 'complex number'.

$$e^{i\pi} + 1 = 0$$

As many have observed, there is something magical about how such a concise expression can harmonise the fundamental concepts 0, 1, e, i, and π - and arithmetic, geometry, algebra, analysis, and real [0,1], complex [i], and transcendental [e, π] numbers.

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Cantor, infinity, and set theory

Cantor (1845-1918) was the creator of modern *set theory* in the 1870s and 1880s. His family background was one of 'civic participation, commercial success, musical accomplishment, and philosophical interests' (Yandell). His first work was on problems concerning infinite processes. He studied under Weierstrass who, building on the work of Cauchy, was trying to put calculus on a firm logical foundation, with proper definitions for 'limit' and sound methods for infinite series. It was this that led Cantor to problems concerning sets, infinity and what Leibniz called 'the labyrinth of the continuum': the number line seems to be made up of an infinity of discrete dimensionless points which correspond to real numbers but this idea produces many paradoxical thoughts.

Galileo discussed the question of greater or lesser infinite sets in *Dialogue Concerning the New Sciences* (1636). He drew attention to the fact that the set of squares was a subset of the set of natural numbers yet was also infinite. Consider the two series of numbers (a) and (b) stretched out infinitely:

(a) 1,2,3,4,5…
(b) 2,4,6,8,10…

(a) is the set of *positive integers* and (b) is the set of *even positive integers*; (b) is a 'proper subset' of (a). If a set S is finite, then a proper subset of S must contain at least one fewer elements than S therefore could not be put into 1-1 correspondence with S. In the world of the infinite, however, the rules seem different. (b) consists of only half the number of terms of (a), but the two different infinite sets *can be put into 1-1 correspondence with each other*, stretching out infinitely far.

The implication of the *Dialogue* is that although it seems that the set of all natural numbers is larger than the set of all squares, paradoxes and problems render the distinction moot: 'the attributes "equal", "greater", and "less" are not applicable to infinite, but only to finite quantities' and it makes no sense to say that a longer line contains more points than a shorter – 'each line contains an infinite number [of points]' (Galileo).

Since ancient Greece, we have struggled with seeming paradoxes concerning the concepts of *continuity* and *infinity*. Zeno's paradox has cropped up again and again. There were four arguments attributed to Zeno by Aristotle. The most famous was 'Achilles and the tortoise': the fast-pursuing Achilles must always come first to the point from which the slow hare has just left, and so on *ad infinitum*, so the slower must always be ahead even as his lead shrinks. There was also the paradox of the Arrow: a moving arrow must at any moment occupy a fixed space but then it must be at rest, it cannot be moving. All of the many formulations raise the same issues: is the number line continuous or discrete, how to reconcile infinite series with finite space and time, and so on.

See Dantzig p.126ff for a discussion of Zeno and modern answers to his problems. They were a huge problem for the development of Greek mathematics. In response, Euclid's severe and rigorous axiomatic geometry dominated. The infinite became a taboo.

> '[T]he development of infinite processes, which had reached quite an advanced stage in pre-Platonic times, was almost completely arrested. We find in classical Greece a confluence of most fortunate circumstances: a line of geniuses of the first rank, Eudoxus, Aristarchus, Euclid, Archimedes, Apollonius, Diophantus, Pappus; a body of traditions which encouraged creative effort and speculative thought and at the same time furthered a critical spirit…; and finally, a social structure most propitious to the development of a leisure class, providing a constant flow of thinkers, who could devote themselves to the pursuit of ideas without regard to immediate utility - a combination of circumstances, indeed, which is not excelled even in our own day. Yet Greek mathematics stopped short of an algebra in spite of a Diophantus, stopped short of an analytic geometry in spite of an Apollonius, stopped short of an infinitesimal analysis in spite of an Archimedes. I have already pointed out how the absence of a notational symbolism thwarted the growth of Greek mathematics; the *horror infiniti* was just as great a deterrent.' Dantzig, p. 133.

Modern mathematics developed concepts to cope with these issues, such as *limit* and *convergence* that show how, for example, an infinite series can sum to a finite number and *calculus*, invented by Newton and Leibniz in the 17th Century, proved to be a practical method of using infinite processes to solve a vast array of problems. However, in many ways the successes of calculus provoked deeper worries about the nature of the infinite and the foundations of mathematics.

Algebraic geometry, traditionally ascribed to Descartes, relies on the idea that it is possible to represent all points on a line, and therefore all points in a plane, by numbers. We know from of the existence of the irrationals that we cannot establish a perfect correspondence between the rational numbers and the points on the number line. What happens when we extend beyond the rational numbers? Can any *real* number be represented by a point on the number line, and can a real number be assigned to *any* point on the number line? If the answer is Yes, then 'we can confidently use the intuitive language of geometry in the formulation of arithmetical analysis' (Dantzig). We naturally think of time and a line as prototypes of *continuous* processes. For the concept of number, it is necessary to think of the number line as a *discrete* succession of infinitesimal points but 'this is repugnant to the very idea of motion conceived by us as the direct opposite of rest' (Dantzig). These problems are necessarily hard for human intuition to cope with.

Cantor found new approaches to these problems. He thought of the point on the number line as a *limiting position in an infinite process applied to a segment of a line* and he used the *rational infinite series* as the basis for his rigorous definition of the *irrationals*.[15]

> 'Our belief in the continuity of the universe and our faith in the causal connection between its events are but two aspects of this primitive intuition that we call time. And so on the one hand there is the conviction that *Natura non facit saltus [nature does not make jumps]*, and on the other hand arises the illusion: *post hoc, ergo propter hoc [after this, therefore because of this]*.

[15] As has often happened in the history of mathematics and science, such as with the near-simultaneous development of calculus by Newton and Leibniz or the Pascal-Fermat discovery of the principles of probability, in parallel to Cantor another mathematician, Richard Dedekind, was developing a different approach to the same problem ('*Continuity and Irrational Numbers*', 1872). His definition of number as a *partition* ('Dedekind cuts') is equivalent to Cantor's and I will not go into it here: see Dantzig p.177ff.

'Herein I see the genesis of the conflict between geometrical intuition, from which our physical concepts derive, and the logic of arithmetic. The harmony of the universe knows only one musical form - the *legato*; while the symphony of number knows only its opposite - the *staccato*. All attempts to reconcile this discrepancy are based on the hope that an accelerated *staccato* may appear to our senses as a *legato*. Yet our intellect will always brand such attempts as deceptions and reject such theories as an insult, as a metaphysics that purports to explain away a concept by resolving it into its opposite.

'But these protests are in vain. To bridge the chasm between the continuity of our concept of time and the inherent discontinuity of the number structure, man had to invoke once more that power of his mind which knows itself capable of conceiving the indefinite repetition of the same act when once this act is possible. This was the historic role of the infinite; this is why through the ages problems of the continuum and of the infinite were but the two horns of a single dilemma.

'This long process of adaptation has now culminated in Cantor's theory in which any number is conceived as the *goal* of an infinite succession of jumps, and the continuum is regarded as comprising not only all possible *resting stations* but all possible goals as well. It is a *staccato* theory *par excellence*; and yet it does not escape the tyranny of time. It merely adapts itself to this tyranny by complacently regarding the flowing stream of duration as an infinite succession of pulsations of furiously accelerated tempo…

'On the one hand, there was the logically consistent concept of the real number and its aggregate, the arithmetic continuum; on the other hand, the vague notions of the point and its aggregate, the linear continuum. All that was necessary was to declare the identity of the two…: *It is possible to assign to any point on a line a unique real number, and, conversely, any real number can be represented in a unique manner by a point on the line.* This is the famous Dedekind-Cantor axiom…

'This axiom, like so many others, is really a disguised definition: it defines a new mathematical being, the arithmetical line. Henceforth the line - and consequently the plane and space - ceases to be an intuitive notion and is reduced to being *a mere carrier of numbers*.

'And so this axiom is tantamount to an arithmetization of geometry. It means the emancipation of analysis from geometrical intuition, to which it owed its birth and growth. It is a bold declaration that henceforth analysis proposes to assume control over geometry and mechanics, and through them to control those other phases of our cognition, which are even nearer to the crude reality of our senses. The age-old struggle to create an arithmetic in the image of that reality had failed because the vagueness of that reality. So arithmetic created a new reality in its own image. The infinite process succeeded where the rational number had failed.' Dantzig, p. 176ff.

In pondering these problems, Cantor created the language of *modern set theory* and sparked a revolution in the foundations of mathematics.

Cantor spent a decade trying to prove that all infinite sets *can* be put into 1-1 correspondence. A set (finite or infinite) is 'denumerable' (or 'countable') if it can be put into 1-1 correspondence with the positive integers. It is 'non-denumerable' (or 'uncountable') if it is infinite but cannot be put into 1-1 correspondence with the positive integers.

One of his first discoveries was that *the set of rational numbers is equivalent to the set of integers*. This immediately seems odd. The rational numbers are dense on the number line (there is an infinite number of them between any two rational numbers) therefore there is no 'next larger' rational number and one cannot arrange the rational numbers in order of size (as with the integers). However, Cantor showed that one can nevertheless put the two sets into 1-1 correspondence. Although the set of rational numbers contains more elements than the set of integers, they are *equivalent* (just as the set of positive integers contains more elements than the set of even positive integers yet they are equivalent). (Courant, p. 79ff).

Cantor first found his proof of *the denumerability of the algebraic numbers and the non-denumerability of the real numbers* in December 1873, and published it in early 1874 in a paper called '*On a Property of the Set of All Algebraic Numbers'*.[16] This was the first proof that *not all infinite sets are the same size*: the reals are equivalent to the points in a (finite or infinite) line, they are a larger infinity than the integers, and are said to have the 'power of the continuum.'[17] In 1891, he would publish a new and much simpler proof of the non-denumerability of the reals using his *diagonal method*.

In 1877-8, Cantor established the 'invariance of dimension'. The uncountability of the continuum does *not* arise from the fact that is infinite in length. 'The entire number continuum is equivalent to any finite segment … [and] even a finite segment of the number axis contains a non-denumerable infinity of points' (Courant, p. 82). Galileo had been right: 'the longer line contains no more points than the shorter'. Further, the fact that an area can be mapped 1-1 onto the continuum means that the complex domain does not have a greater power than the continuum of a one dimensional line.

Between 1879-1884, Cantor published six papers which founded his new *set theory*, of which the fifth, '*Grundlagen*' (1883) was the most important. Cantor defined a set as a collection of definite, distinguishable 'objects of perception or thought' that were conceived as a whole entity. $x \in A$ [x is a member of set A] iff [iff means 'if and only if'] x meets the conditions of S(x) which defines members of the set (the principle of abstraction). The number of elements in a set is its *cardinal* number. To say that a set A has n elements means that A can be put into 1:1 correspondence with the set of integers 1 to n. A is a proper subset of B if A lacks at least one element of B. There is only one empty set; it is a subset of every set. The set of all subsets of A is the 'power set' of A; if A is finite with *n* elements, the number of subsets (including the empty set) is 2^n (e.g. the number of subsets of a set with 3 elements is $2^3=8$ subsets).

In 1883, he *introduced 'transfinite' numbers* to describe the different types of infinite set. ℵ0 (*aleph-null*) is the first transfinite cardinal number and is the 'cardinality' of the infinite set of natural numbers.

Grundlagen also introduced the *Continuum Hypothesis* which would be top of Hilbert's famous 1900 list of outstanding great mathematical problems to be solved and which engaged so much of Gödel's efforts. The Continuum Hypothesis (CH): there is no infinite set that has more elements than there are positive integers, and that has fewer elements than there are real numbers; i.e there is no set with a cardinality between the set of integers and the set of reals.

Cantor believed that his Hypothesis was true and would be provable. After the development of axiomatic set theory (see below), usually referred to as Zermelo-Frankel set theory with 'the axiom of choice' (or 'ZFC'), in response to the problems with paradoxes of set theory, Gödel

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[16] It seems he chose the title to de-emphasise the real point of the paper, the non-denumerability of the reals, in order to minimise opposition from Kronecker. The paper also furnished an additional corroboration of Liouville's proof that there are infinitely many transcendentals between any two intervals (Dauben).

[17] His method involved defining an algebraic equation's 'height' as the sum of the absolute values of coefficients plus its degree minus 1. So: $2x^3 - 3x^2 + 4x - 5 = 0$ has a height of 2+3+4+5+(3-1)=16. He proved that this allowed him to order all algebraic equations and thereby prove they are denumerable.

would later prove (1940) that adding the *Continuum Hypothesis* as an axiom to ZFC is consistent if ZFC itself is consistent, then Cohen would prove (1963) that adding an axiom that there *is* an intermediate set (i.e. the CH is false) is also consistent with ZFC. (ZFC itself is, as fas as we know, consistent; see Franzen below on misunderstandings of Gödel.) That is, the CH is 'independent' of ZFC: 'You can add a new axiom asserting there is a set with intermediate power, or that there is no such set, and the resulting system of axioms will not lead to a contradiction' (Chaitin, 2006). For *formalists*, the Gödel-Cohen proofs suggest that the CH is neither true nor false, it just depends what formal axiomatic system one chooses to work with. For *Platonists* like Gödel, the CH is true or false independent of the choice of formal axiomatic system. Gödel himself thought the CH is false.

In *On Linear Aggregates* (1883), Cantor explained that henceforth maths should treat the infinite as something definite and not just in the sense of 'limits' developed with calculus.

> 'It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, which it acquired during the 17th century. I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulations, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as a logical necessity…
>
> 'The existence of the infinite will never again be deniable while that of the finites is nevertheless upheld. If one permits either to fall, one must do away as well with the other.'

He rejected ancient arguments such as Aristotle's that admitting infinities means 'annihilating' normal finite numbers (e.g. If *a + inf = inf*, then *a* is 'annihilated') on the grounds that one could not expect transfinite numbers to work on the same arithmetical rules as the normal numbers, and the irrational numbers also owe their acceptance by virtue of infinite rational sets.

Conceiving of the infinite as something definite and developing an arithmetic of transfinites was a profound revolution. The prevailing attitude towards these 'traditions' had been expressed by Gauss in a letter of 1831.

> 'As to your proof, I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a figure of speech; an abridged form for the statement that limits exist which certain ratios may approach as closely as we desire, while other magnitudes are permitted to grow beyond all bounds… No contradictions will arise as long as Finite Man does not mistake the infinite for something fixed, as long as he is not led by an acquired habit of mind to regard the infinite as something bounded.'

Cantor was proposing that Finite Man *should* regard the infinite as 'something fixed'.

From 1884, Cantor was increasingly unhappy about his inability to prove the Continuum Hypothesis and this unhappiness was exacerbated by his feud with Kronecker. In 1884, he had his first major breakdown and after recovering he spent a lot of time on other subjects including the authorship of Shakespeare's plays. From then, he had more frequent and severe attacks of depression and paranoia and sadly, like Gödel after him, he developed paranoid ideas about conspiracies and persecution.

In 1891, after struggling since 1884 with his health and motivation, Cantor presented a new paper that contained his famous *'diagonal proof'* of the non-denumerability of the real numbers.

Imagine we *can*, somehow, list all real numbers one after the other down the page. Imagine such a list…

1. 0.**2**462…
2. 0.8**3**83…
3. 0.02**8**5...
4. … and so on down the page…

However, we can easily produce a number that we know cannot be on any such list. Look at the first digit of the first number (2) and start our new number with any other digit, say 5. For the second digit, pick any number but 3, say 2. For the third digit, avoid 8, and so on. We know as we go on that the new number will be different from the first, second, and so on numbers listed; our new number is not on the list, and if we add it we can simply repeat the process to produce another entirely new number. It does not matter that the list goes on infinitely; it *cannot* contain an infinite list of all possible numbers and therefore it cannot be put into 1-1 correspondence with the set of natural numbers. Cantor's *diagonal method* would later be used by Gödel and Turing.

This paper also showed that for any given set S, the set of its subsets is always of a greater power than S itself and this in turn shows that there is a 'forever increasing succession of transfinite cardinal numbers, the powers of transfinite sets'. *For any set A, it is impossible to put A into 1-1 correspondence with the power set of A (i.e the set of all subsets of A)*; that is, the cardinality of the power set of A is greater than the cardinality of A. *The set of all subsets of the natural numbers is non-denumerable and there are infinitely many sizes of infinite sets*; there is no last transfinite number, just as there is no last finite number. The set of all subsets of the natural numbers is the same size as the set of all points on a straight line - the *continuum* - which is the same size as the set of all reals.

Cantor was no nearer proving the Continuum Hypothesis but in his continued attempts to prove it using his new set theory he first discovered some *paradoxes* in set theory some time in 1895 (see below). Set theory was immensely controversial. Kronecker, Cantor's old teacher and tormentor, famously said that 'God made the integer, the rest is the work of man.' He attacked Cantor as a 'corrupter of youth' and tried to suppress publication of Cantor's work and block his academic advancement. Although set theory would be described as a 'paradise' by Hilbert, Poincaré was later thought to have said that, 'Later generations will regard set theory as a disease from which one has recovered' though Gray says this widely quoted comment is in fact 'entirely erroneous'.[18] Wittgenstein would denounce Cantor's 'pernicious idioms', though his views on maths were not taken seriously by top mathematicians such as Gödel and von Neumann. Weyl would warn that, 'We have stormed the heavens, but succeeded only in building fog upon fog, a mist which will not support anybody who earnestly desires to stand upon it.'

However, the immense usefulness of calculus, which is built upon infinite processes, meant that 'the empire' of calculus kept expanding even while the top mathematicians argued about its logical basis (Dantzig).[19]

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[18] Gray p.86 (2000).

[19] At the same time as Cantor was working on his ideas, Dedekind published his ideas on the continuum and infinity and the two sets of ideas were shown to be equivalent. Cf. Dantzig's brilliant book for an explanation.

Frege (1848-1925), logic, and formal axiomatic systems

'For the infinite will eventually refuse to be excluded from arithmetics...Thus we can foresee that this issue will provide for a momentous and decisive battle.' Frege.

In 1879, Frege published *Begriffsschrift*, translated as 'concept writing' or 'concept script'.[20] Whereas Aristotle's logic could not deal with even very simple mathematics, Frege essentially created what is now known as '*first order logic*' (a subset of what is often referred to as 'predicate logic', or 'predicate calculus'). It is distinguished from the earlier and simpler *propositional logic* partly by its use of two *quantifiers,* the *existential quantifier* (∃) and the *universal quantifier* (∀).[21]

Frege's purpose was to construct a *formal axiomatic system* in which all rules of inference are set out explicitly, hence results could be proved mechanically without reference to intuition. It was a new attempt to develop Leibniz's system for mechanising thought (see previous blog HERE).

> 'If the task of philosophy is to break the domination of words over the human mind [...], then my concept notation, being developed for these purposes, can be a useful instrument for philosophers.'

Frege was also trying to reduce all mathematics to logic and show that it is *analytic* as Hume had argued *contra* Kant. Frege's work was therefore the catalyst for what became known as 'analytical philosophy'. Russell would later write:

> 'From Frege's work, it followed that arithmetic, and pure mathematics generally, is nothing but a prolongation of deductive logic. [Not necessarily!] This disproved Kant's theory that arithmetical propositions are 'synthetic' and involve a reference to time. [Not necessarily!]' (See below re Russell.)

Frege agreed with the results of Cantor's transfinite arithmetic but he strongly opposed Cantor's methods which he believed were not at all strong enough to provide a sure foundation for set theory and mathematics. He strongly criticised Cantor's definition of 'set'. In 1892 he warned of the dangers of not providing an absolutely rigorous logical basis for Cantor's ideas of the infinite.

> 'Here is the reef on which it [mathematics] will founder. For the infinite will eventually refuse to be excluded from arithmetic, and yet it is irreconcilable with that [finitist] epistemological direction. Here, it seems, is the battlefield where a great decision will be made.'

Frege was confident that his logical system would provide the means to integrate infinity into mathematics in a consistent way and thereby end much of the recent arguments concerning the continuum and infinity. However, his confidence that his system was consistent and free from paradoxes would be famously blown up by the letter he received from Russell in 1902.

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Poincaré, 'mathematical induction', infinity, and proof

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[20] Its full title is something like '*Concept script, a formal language, modelled on arithmetic, of pure thought*'.

[21] Frege's First-Order Logic was based on: objects (e.g. 1); predicates, or relationships (e.g. 1<2); operations, or functions (e.g. 1+2); logical operations (e.g. AND); quantifiers (e.g. ∃ and ∀).

What constitutes proof? Is *formal logic* sufficient for mathematical proofs and if not how do we cope with the fact that it therefore seems to rest upon *intuition*? There were two responses: *formalism* and *intuitionism*. Hilbert was the champion of the former, Poincaré of the latter.[22]

In mathematics we have the *deductive* model, since Aristotle at least:

Premises (axioms, definitions, postulates) + logic = logical or mathematical *deduction* (e.g. geometry). Deduction is 'based on the principle of contradiction and on nothing else' (Dantzig).

In the natural sciences we have the *inductive* model:

Observation + hypothesis + test = scientific *induction*. This model, scientific induction, has no validity for mathematical proof.

Unfortunately the phrase '*mathematical induction*' confuses the terminology. Dantzig urges that instead we use the term '*reasoning by recurrence*'. Whatever one calls it, what is it?

1. Is the proposition we wish to demonstrate *hereditary*, i.e. if it is true for any member of a sequence it is necessarily true for the successor of the member as a logical necessity?
2. Is the proposition true for the first term of the sequence?

There are many examples of famous mathematical hypotheses that held for large numbers but finally someone found some counter-example. Mathematical proofs need to establish that even if one keeps going infinitely long one could *never* find a counter-example.

The principle of recurrence was first explicitly formulated by Pascal in his *The Arithmetic Triangle* (1654) and in his famous correspondence with Fermat regarding gambling.

> 'It is surely a fitting subject for mystic contemplation, that the principle of reasoning by recurrence, which is so basic in pure mathematics, and the theory of probabilities, which is the basis of all inductive sciences, were both conceived while devising a scheme for the division of the stakes in an unfinished match of two gamblers.' Dantzig

In his *The Nature of Mathematical Reasoning* (1894), Poincaré posed a fundamental problem: *either maths is not purely deductive, in which case its rigour is in doubt, or it is entirely deductive in which case it is just 'an immense tautology'.*

> 'The very possibility of a science of mathematics seems an insoluble contradiction. If this science is deductive only in appearance, whence does it derive that perfect rigour which no one dares to doubt? If, on the contrary, all the propositions it enunciates can be deduced one from the other by the rules of formal logic, why is not mathematics reduced to an immense tautology? The syllogism can teach us nothing that is essentially new, and, if everything is to spring from the principle of identity, everything should be capable of being reduced to it. **Shall we then admit that the theorems which fill so many volumes are nothing but devious ways of saying that A is A?** [Emphasis added.]
>
> 'No doubt we may refer back to axioms which are at the source of all these reasonings. If it is felt that they cannot be reduced to the principle of contradiction, if we decline to see in

[22] Poincaré defined 'objective reality' as 'what is common to many thinking beings and could be common to all' which is a similar idea to Kant's *sensus communis*. In Dantzig's words, this definition of objective reality 'is a fundamentally subjective consensus of what is commonly held, or what could be held, to be objective.'

them any more than experimental facts which have no part or lot in mathematical necessity, there is still one resource left to us: we may class them among *a priori* synthetic views. But this is no solution of the difficulty - it is merely giving it a name… Syllogistic reasoning remains incapable of adding anything to the data that are given it; the data are reduced to axioms, and that is all we should find in the conclusions.

'No theorem can be new unless a new axiom intervenes in its demonstration… Should we not therefore have reason for asking if the syllogistic apparatus serves only to disguise what we have borrowed?…

'Finally, if the science of number were merely analytical, or could be analytically derived from a few synthetic intuitions, it seems that a sufficiently powerful mind could with a single glance perceive all its truths; nay, one might even hope that some day language would be invented simple enough for these truths to be made evident to any person of ordinary intelligence.

'… [I]t must be granted that mathematical reasoning has of itself a kind of creative virtue, and is therefore to be distinguished from the syllogism. We shall not find the key to the mystery in the frequent use of the rule by which the same uniform operation applied to equal numbers will give identical results. All these modes of reasoning, whether or not reducible to the syllogism, properly so called, retain the analytical character, and *ipso facto*, lose their power.'

Poincaré distinguishes between *proof* and *verification*.

'Verification differs from proof precisely because it is analytical and because it leads to nothing. It leads to nothing because the conclusion is nothing but the premises translated into another language. A real proof is fruitful because the conclusion is more general than the premises… If mathematics could be reduced to a series of such verifications [i.e. just step by step deductions from axioms] it would not be a science. A chess-player, for instance, does not create a science by winning a piece. There is no science but the science of the general. It may even be said that the object of the exact sciences is to dispense with these direct verifications.'

Poincaré explains why *the principle of recurrence* is required for maths to make general statements 'applicable to all numbers, which alone is the object of science'. Without it 'we should require an infinite number of syllogisms, and we should have to cross an abyss which the patience of the analyst, restricted to the resources of formal logic, will never succeed in crossing.'

'The essential characteristic of *reasoning by recurrence* is that it contains in a single formula an infinite number of syllogisms [i.e. the theorem is true of 1, if true of 1 then it is true of 2… *ad infinitum*]…

'[In arithmetic the mathematician] cannot conceive its general truths by direct intuition alone; **to prove even the smallest theorem he must use reasoning by recurrence, for that is the only instrument which enables us to pass from the finite to the infinite**. This instrument … frees us from the necessity of long, tedious, and monotonous verifications which would rapidly become impracticable… In this domain of Arithmetic we may think ourselves very far from the infinitesimal analysis, but the idea of mathematical infinity is already playing a preponderating part, and without it there would be no science at all, because there would be nothing general.'

The rule of reasoning by recurrence is irreducible to the principle of contradiction and cannot come to us from experiment - it is 'the exact type of the *a priori synthetic* intuition'. The mind has 'a direct intuition [that] it can conceive of the indefinite repetition of the same act, when the act is once possible'. While induction in the physical sciences 'is always uncertain, because it is based on the belief in a general order of the universe, an order which is external to us', proof by recurrence is 'necessarily imposed on us, because it is only the affirmation of a property of the mind itself.'

> 'But we shall always be brought to a full stop - we shall always come to an indemonstrable axiom,[23] which will at bottom be but the proposition we had to prove translated into another language. We cannot therefore escape the conclusion that the rule of reasoning by recurrence is irreducible to the principle of contradiction. Nor can the rule come to us from experiment. Experiment may teach us that the rule is true for the first ten or the first hundred numbers, for instance; it will not bring us to the indefinite series of numbers, but only to a more or less long, but always limited, portion of the series.
>
> 'Now, if that were all that is in question, the principle of contradiction would be sufficient, it would always enable us to develop as many syllogisms as we wished. It is only when it is a question of a single formula to embrace an infinite number of syllogisms that this principle breaks down, and there, too, experiment is powerless to aid. This rule, inaccessible to analytical proof and to experiment, is the exact type of the *a priori synthetic* intuition. On the other hand, we cannot see in it a convention as in the case of the postulates of geometry.
>
> 'Why then is this view imposed upon us with such an irresistible weight of evidence? It is because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it, and thereby of becoming conscious of it.
>
> 'But it will be said, if the legitimacy of reasoning by recurrence cannot be established by experiment alone, is it so with experiment aided by induction? We see successively that a theorem is true of the number 1, of the number 2, of the number 3, and so on - the law is manifest, we say, and it is so on the same ground that every physical law is true which is based on a very large but limited number of observations.
>
> 'It cannot escape our notice that here is a striking analogy with the usual processes of induction. But an essential difference exists. Induction applied to the physical sciences is always uncertain, because it is based on the belief in a general order of the universe, an order which is external to us. Mathematical induction - i.e., proof by recurrence - is, on the contrary, necessarily imposed on us, because it is only the affirmation of a property of the mind itself…
>
> 'No doubt mathematical recurrent reasoning and physical inductive reasoning are based on different foundations, but they move in parallel lines and in the same direction - namely, from the particular to the general.'

Poincaré then concludes that it is only the principle of recurrence that allows us to 'learn something new':

> 'Mathematicians therefore proceed "by construction," they "construct" more complicated combinations. When they analyse these combinations, these aggregates, so to speak, into their

[23] Why? Is this an odd foreshadow of Gödel's proof?

primitive elements, they see the relations of the elements and deduce the relations of the aggregates themselves. The process is purely analytical, but it is not a passing from the general to the particular, for the aggregates obviously cannot be regarded as more particular than their elements.

'Great importance has been rightly attached to this process of "construction," and some claim to see in it the necessary and sufficient condition of the progress of the exact sciences. Necessary, no doubt, but not sufficient! For a construction to be useful and not mere waste of mental effort, for it to serve as a stepping-stone to higher things, it must first of all possess a kind of unity enabling us to see something more than the juxtaposition of its elements. Or more accurately, there must be some advantage in considering the construction rather than the elements themselves. What can this advantage be?…

'A construction only becomes interesting when it can be placed side by side with other analogous constructions for forming species of the same genus. To do this we must necessarily go back from the particular to the general, ascending one or more steps. The analytical process "by construction" does not compel us to descend, but it leaves us at the same level. We can only ascend by [the principle of recurrence], for from it alone can we learn something new. Without the aid of this [mathematical] induction [i.e. the principle of recurrence], which in certain respects differs from, but is as fruitful as, physical induction, construction would be powerless to create science.'

The restricted principle of recurrence is, Dantzig concludes, 'logically unassailable' when applied to finite sequences; 'it is a consequence of classical logic.' However, the general principle of recurrence used in arithmetic 'tacitly asserts that any number has a successor'.

'This assertion is not a logical necessity, for it is not a consequence of the laws of classical logic. This assertion does not impose itself as the only one conceivable, for its opposite, the postulation of a finite series of numbers, leads to a bounded arithmetic which is just as tenable. This assertion is *not* derived from the immediate experience of our senses, for all our experience proclaims its falsity. And finally this assertion is *not* a consequence of the historical development of the experimental sciences, for all the latest evidence points to a bounded universe, and in the light of the latest discoveries in the structure of the atom, the infinite divisibility of matter must be declared a myth. And yet the concept of infinity, though not imposed upon us by logic or by experience, is a *mathematical necessity*.' (Dantzig)

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Hilbert, axiomatisation, and his 1900 speech setting the agenda for future maths

'No one shall expel us from the Paradise that Cantor has created.' Hilbert

Hilbert, unlike many world famous mathematicians, was not a child prodigy. He did not seem to have a great memory and struggled with languages. Unlike many other European academics, he did not succumb to the nationalist furore of the age; when sitting next to the new Nazi education minister, he bravely replied to a question about the state of mathematics in Göttingen now that 'the Jewish influence' had gone - 'Mathematics in Göttingen? There really isn't any any more.'

Before the 19th Century, Euclidean geometry was regarded as logically flawless with clear axiomatic foundations while calculus was regarded as suffering an embarrassing lack of logically secure foundations. The 19th Century reversed this. Non-Euclidean geometry, discovered by Gauss *et al*, exploded two thousand years of assumptions and broke Kant's view that Euclidean geometry is an example of the *synthetic a priori*. (Einstein would realise, prompted by Grossmann, that he needed non-Euclidean geometry for General Relativity.) Meanwhile Cauchy, Weierstrass, Dedekind and Cantor built a logical foundation for calculus, though one based on taming infinity.

Hilbert decided early to study these foundational issues in mathematics. For part of his thesis defence, he chose to argue that 'objections to Kant's theory of the *a priori* nature of arithmetical judgements are unfounded'. In 1899, Hilbert produced a new axiomatisation of Euclidean geometry (*Grundlagen der Gerometrie)* in which geometry was reduced to a purely logical treatment based on axioms. It no longer depended on any particular visual representation and was logically separated from empirical concepts, or visual intuitions of space, which had misled mathematicians about the logical status of Euclid's axioms. Although he used words like 'point' and 'line', the symbolic logic of the axioms was the true foundation and Hilbert would famously say that instead of words like point and line one could easily substitute 'tables, chairs, and beer mugs'.

His axiomatic system for Euclidean geometry was based on the assumption that real number arithmetic is *consistent*. A point is a pair of real numbers and a line is a set of pairs of numbers that satisfy the equation of a line. Euclid's axioms therefore are translated into true statements about real numbers and Euclidean geometry 'is reduced to a fraction of all the true statements about real numbers' (Yandell). He then showed how non-Euclidean geometries could be axiomatised; they are consistent if his axioms for Euclidean geometry are consistent, which in turn depend on the consistency of real number arithmetic. In the second edition of *Grundlagen*, Hilbert introduced the idea that an axiomatic system could be *complete*. For Hilbert, one could have a *consistent* set of axioms for both Euclidean and non-Euclidean geometry.[24] These concepts of *completeness* and *consistency* would prove absolutely fundamental. Any doubts about the consistency of the number system would undermine the basis of his axiomatisation of geometry.

In 1900, Hilbert set out 23 mathematical questions to be solved in a famous lecture, *Mathematical Problems*,[25] that enormously influenced 20th Century mathematics. Attacking these problems has created vast new fields of knowledge even if the problem itself has not been conquered.

He described the connection between reason and experience, between mathematics and physics:

> 'Having now recalled to mind the general importance of problems in mathematics, let us turn to the question from what sources this science derives its problems. Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. Even the rules of calculation with integers must have been discovered in this fashion in a lower stage of human civilization, just as the child of today learns the application of these laws by empirical methods. The same is true of the first problems of geometry, the problems bequeathed us by antiquity, such as the duplication of the cube, the squaring of the circle; also the oldest problems in the theory of the solution of numerical equations, in the theory of curves and the differential and integral calculus, in the calculus of variations, the theory of Fourier series and the theory of potential - to say nothing of the further abundance of problems properly belonging to mechanics, astronomy and physics.

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[24] For Frege, 'there is only one world and so only one geometry, and ... non-Euclidean geometry ... is simply meaningless... In fact, mathematicians had proved that if one accepts one of these geometries one must accept the other (they are ... relatively consistent).... This shows that Frege had no grasp at all of the relative consistency argument.' (Gray).

[25] For an English translation of Hilbert's (1900) cf.: http://aleph0.clarku.edu/~djoyce/hilbert/problems.html. In his actual spoken lecture he only discussed 10 Problems; the others were added in the published version.

> 'But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalisation, specialisation, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. Thus arose the problem of prime numbers and the other problems of number theory, Galois's theory of equations…; indeed almost all the nicer questions of modern arithmetic and function theory arise in this way.
>
> 'In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience.'

He then asks what 'general requirements may be justly laid down for the solution of a mathematical problem':

> '… that it shall be possible to establish the correctness of the solution by means of a finite number of steps based upon a finite number of hypotheses which are implied in the statement of the problem and which must always be exactly formulated. This requirement of logical deduction by means of a finite number of processes is simply the requirement of rigour in reasoning.'

He then points out that a 'solution' is often a proof that a solution is impossible, for example, for the famous three ancient problems such as squaring the circle, and states his famous belief that '*in mathematics there is no ignorabimus*'.

> 'It is probably this important fact along with other philosophical reasons that gives rise to the conviction (which every mathematician shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts.
>
> 'Take any definite unsolved problem, such as … the existence of an infinite number of prime numbers of the form $2^n + 1$. However unapproachable these problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes.
>
> 'Is this axiom of the solvability of every problem a peculiarity characteristic of mathematical thought alone, or is it possibly a general law inherent in the nature of the mind, that all questions which it asks must be answerable? For in other sciences also one meets old problems which have been settled in a manner most satisfactory and most useful to science by the proof of their impossibility. I instance the problem of perpetual motion. After seeking in vain for the construction of a perpetual motion machine, the relations were investigated which must subsist between the forces of nature if such a machine is to be impossible; and this inverted question led to the discovery of the law of the conservation of energy, which, again, explained the impossibility of perpetual motion in the sense originally intended.

> 'This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus* ["*Ignoramus et ignorabimus*", We do not know and we shall never know].'

The *First Problem* was Cantor's *Continuum Hypothesis*, which Gödel and Cohen would solve (see above). Hilbert defined his *Second Problem*, *The Compatibility of the Arithmetical Axioms*, the problem Gödel would answer in 1931, as a proof that the axioms of arithmetic are not contradictory and that '*a definite number of logical steps based upon them can never lead to contradictory results*.'

> 'When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the definitions of those elementary ideas; and no statement within the realm of the science whose foundation we are testing is held to be correct unless it can be derived from those axioms by means of a finite number of logical steps…
>
> 'I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: *To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results…*
>
> 'In geometry, the proof of the compatibility of the axioms can be effected by constructing a suitable field of numbers, such that analogous relations between the numbers of this field correspond to the geometrical axioms. Any contradiction in the deductions from the geometrical axioms must thereupon be recognizable in the arithmetic of this field of numbers. In this way the desired proof for the compatibility of the geometrical axioms is made to depend upon the theorem of the compatibility of the arithmetical axioms.
>
> '*On the other hand a direct method is needed for the proof of the compatibility of the arithmetical axioms*. [Emphasis added] The axioms of arithmetic are essentially nothing else than the known rules of calculation, with the addition of the axiom of continuity… I am convinced that it must be possible to find a direct proof for the compatibility of the arithmetical axioms, by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers.'

Hilbert's 1900 speech was confident. However, the foundations of the whole structure were tottering.

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The discovery of paradoxes in set theory, Russell's attempted cure, the 'logical positivists'

Cantor first discovered some *paradoxes* in his set theory some time in 1895. He sent a letter to Hilbert about it in 1896. Hilbert, who was then working on his axiomatisation of Euclidean geometry, kept quiet about Cantor's problem (why?). Cantor seems not to have discussed the issue until 1899 when he wrote to Dedekind about it. In 1897, Burali-Forti published a paper on the paradoxes of set theory but it got little attention.

In 1902, just as Frege was about to finish the second volume of his *Basic Laws of Arithmetic*, he got a letter from Russell pointing out what would later become known as *Russell's paradox*: what about a *set (S) of sets that are not members of themselves*? If S is *not* a member of itself, then it is 'a set that is not a member of itself', therefore it *should* be a member of S. But if S *is* a member of itself, then it contradicts its own definition of 'a set that is not a member of itself'. (A more colloquial formulation is: a barber shaves all those who do not shave themselves (set S); is the barber a member of the set S? Either a yes or no answer gives a contradiction.) Dantzig wrote that all the paradoxes seem to rest on one question: why can't we have an aggregate of all aggregates, and if so *there is a last transfinite* number, yet this contradicts Cantor's proof that there is no last transfinite number.

Frege wrote in a Postscript of October 1902 that was included in the publication of 1903:

> 'Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter from Mr. Bertrand Russell just when the printing of this volume was nearing its completion…
>
> '*Solatium miseris, socios habuisse malorum*… What is in question is not just my particular way of establishing arithmetic but whether arithmetic can possibly be given a foundation at all.'

In response to these paradoxes, which caused such bewilderment and concern for mathematics, two schools formed. The first was the *formalists* (whose most famous champion was Hilbert), whose response was not to abandon Cantor's set theory but to seek *a complete and consistent axiomatic system for arithmetic* to banish the paradoxes and contain problems arising from the use of infinities. The second was the *intuitionists* (whose most famous champion was Poincaré), who thought this attempt doomed to fail and who regarded Cantor's introduction of infinite sets as the main reason for the disasters.

The problem was later summarised by von Neumann (1947):

> 'In the late nineteenth and the early twentieth centuries a new branch of abstract mathematics, G. Cantor's theory of sets, led into difficulties. That is, certain reasonings led to contradictions; and, while these reasonings were not in the central and 'useful' part of set theory, and always easy to spot by certain formal criteria, it was nevertheless not clear why they should be deemed less set-theoretical than the 'successful' parts of the theory. Aside from the *ex post* insight that they actually led into disaster, it was not clear what *a priori* motivation, what consistent philosophy of the situation, would permit one to segregate them from those parts of set theory which one wanted to save. A closer study of the merits of the case, undertaken mainly by Russell and Weyl … showed that the way in which not only set theory but also most of modern mathematics used the concepts of 'general validity' and of 'existence' was philosophically objectionable.'

There were two main formalist responses to the crisis of paradoxes - Russell's and Zermelo's.

Russell's and Whitehead's *Principia* (published in three volumes 1910-13) was one attempt to rescue Frege's reduction of mathematics to logic. Russell thought that he had at least partially succeeded and even post-Gödel would write that he had cleared up 'two millennia of muddle-headedness about "existence", beginning with Plato's *Theaetetus*'. He had not. Soon, it was clear that *Principia* had *not* provided a complete and consistent formal axiomatic system that solved the problems of paradoxes.

After the publication of Wittgenstein's *Tractatus* in 1921, Russell would soon fall in with Wittgenstein and his circle of admirers - the *logical positivists* of 'the Vienna Circle' including Neurath, Carnap, Hahn (Gödel's dissertation advisor), and Schlick.

Comte (1798-1857), the father of Positivism, had described mankind as developing through a 'law of three phases': the *Theological* (pre-Enlightenment), the *Metaphysical* (post-Enlightenment), and the *Scientific* or *Positive* (post-Napoleon). He envisioned *sociology* as a science of the social world that would replace metaphysics with scientific methodology and this desire caught the later 19th Century *Zeitgeist* as Europe industrialised. Tellingly he had earlier used the phrase 'social physics' but dropped it in favour of the neologism.

The physicist Ernst Mach was influential in the spread of positivist ideas around the turn of the century. His dictum 'where neither confirmation nor refutation is possible, science is not concerned' was influential within and outside the natural sciences, influencing physicists such as the early Einstein and economists such as Schumpeter.[26] Mach's positivism led him to reject atomic theory (where are they?) and Relativity (where's the evidence?).

Strict Positivists reject questions such as 'what is it really?' in favour of asking 'what can I know and prove?' Instead of asking 'what is light?', they ask, 'what can I prove about light?'. The spread of Positivism was connected to a trend in which the methods of natural science were often inappropriately applied to the social sciences, what Hayek later termed 'scientism' - a tendency, for example, to think political or economic problems can be treated as engineering problems.

During the later 19th Century and early 20th Century, Formalism and Positivism invaded most intellectual spheres and became entangled with the general movement that became known as 'Modernism' in the early 20th Century. Picasso, Schoenberg, Freud, Joyce *et al* attacked traditional wisdom and hierarchies in painting, music, psychology, and literature. This *Zeitgeist* involved a rejection of the 'irrational' messiness of liberal capitalism and the injustice that seemed inherent in it, and a growing belief in collectivist claims that society could be managed rationally by the application of 'scientific' methods.

After the war, the 'Vienna Circle' developed what they called *Logical Positivism* out of what they, self-consciously but controversially, thought were the lessons of Wittgenstein's *Tractatus*, an extreme empiricism which regarded 'the true' as synonymous with 'what can be proved' – a dangerous idea and one that Gödel formally disproved with his 1931 paper (see below). For the Vienna Circle, mathematical and logical 'truths' are true only in a formal sense as a result of the rules established for the system: they are *analytical* and *tautological*. In Kant's terms only *a priori analytical* and *a posteriori synthetic* statements are valid and Kant's *synthetic a priori* judgements are outlawed as meaningless. Mathematics was therefore a *syntactic* subject. Its truths derive from the rules of a formal system and these rules are of three basic kinds: (1) the rules for the symbols of the system; (2) the rules for how the symbols can form "well-formed formulas" ("wffs"); (3) the rules of inference that describe how *wffs* can be derived from other *wffs* (Goldstein).

One of the logical positivists praised by Russell was Carnap, one of Wittgenstein's most influential acolytes, who thought that 'all philosophical problems are really syntactical and … when errors in syntax are avoided, a philosophical problem is thereby either solved or shown to be insoluble' (Russell). Carnap did most to stretch Wittgenstein's *Tractatus* to argue that mathematics was also a language system in which questions of truth are merely questions about how you set a system up. Carnap described his view in his *Principle of Tolerance*:

[26] Schumpeter thought Mach's philosophy justified Walras' approach to economics - a system of equations describing a general equilibrium - and argued that economics must focus on measurable quantities.

> 'In logic there are no morals. Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that … he must state his method clearly, and give syntactical rules instead of philosophical arguments.'

The Vienna Circle issued their Manifesto in 1929, *The Scientific Conception of the World* (*Wissenschaftliche Weltauffassung*). The principal author of the Manifesto and one of the most influential positivists was Otto Neurath. Neurath wished to put the social sciences on the same 'scientific' level as the physical sciences. Neurath argued after 1918 that World War I had showed that a fully centralised socialist economy was both possible and necessary. Money should be abolished because it rendered rational planning impossible. Centralised statistical analysis could provide the tools for central planners (cf. Neurath's *Through War Economy to Economy in Kind*, 1919).

In 1931, Neurath wrote:

> 'Of all the attempts at creating a strictly scientific unmetaphysical physicalist sociology, Marxism is the most complete.'

What was *The Scientific Conception of the World*?

> '*First* it is *empiricist and positivist*: there is knowledge only from experience […] *Second*, the scientific world-conception is marked by the application of a certain method, namely *logical analysis*.'

This general approach spilled out from Vienna. Most immediately, it affected other philosophers such as the English philosopher Ayer but the Circle's influence spread far beyond academic philosophers. For example, the behaviourist school of psychology, which wreaked so much harm for so long, was influenced by the Positivists: we should only concern ourselves with what we can observe directly, and not consider what these observations may tell us about, for example, the brain. The Circle's influence was also connected with what they thought were the philosophical lessons of the 1920s revolution in quantum mechanics but I will not go into this aspect here.

Regarding the issue of logic and mathematics, it is crucial that *Gödel and von Neumann dismissed the Vienna Circle as completely wrong* concerning the foundations of mathematics but this did not stop the strong influence of the Circle on inter-war culture. This has led to many misunderstandings. The recent book *Wittgenstein's Poker*, for example, wrongly describes Gödel's 1931 paper as a natural consequence of the 'voice' of the Circle.

Russell would later admit that he had not studied Gödel's 1931 paper and the subsequent arguments. Gödel himself would remark that Russell's *mis*understandings were 'most interesting' unlike Wittgenstein who, Gödel would later write, 'advances a completely trivial and uninteresting misinterpretation' of Gödel's 1931 paper. In a letter of 1958, Gödel wrote of Wittgenstein's book on the foundations of mathematics that its value was that 'it shows the falsity of the assertions set forth in it'. He was also privately dismissive about *Tractatus*.[27] Both Gödel and von Neumann were dismissive of Carnap and did not take either his mathematics or philosophy seriously. Carnap was, wrote von Neumann, a 'turbid source' of ideas, he merely expressed 'completely naïve simplistic views with a terribly important air', and he said of Carnap's views on the implications of Gödel

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[27] An example: in *Tractatus*, Wittgenstein wrote that 'there can *never* be surprises in logic' (6.125). Gödel would shortly prove the opposite - that mathematics *necessarily* surprises.

1931 that Carnap 'obviously has absolutely no understanding of the meaning of Gödel's results' (cf. Leonard, *From Chess to Catastrophe*, 2006).[28]

Although Gödel often pointed out privately the errors of the Circle and Wittgenstein it was not his style to engage in public disputes so many important academics have got a false impression of this subject. For example, in one of Gödel's many unposted letters he wrote to a sociologist who had made the usual claim that his 1931 was part of the 'intellectual atmosphere' of the early 20th Century:

> '[T]he philosophical consequences of my results, as well as the heuristic principles leading to them, are anything but positivistic or empiricistic… I was a conceptual and mathematical realist since about 1925. I never held the view that mathematics is syntax of language. Rather this view, understood in any reasonable sense, can be *disproved* by my results.'

Decades later, he wrote in a letter:

> '… it is indeed clear from the passage you cite that Wittgenstein did *not* understand it [Gödel's 1931 paper]… He interprets it as a kind of logical paradox, while in fact it is just the opposite, namely a mathematical theorem in an absolutely uncontroversial part of mathematics (finitary number theory…) Incidentally, the whole passage you cite seems nonsense to me.'

*

Hilbert's programme and the *Entscheidungsproblem* (decision problem)

Parallel to *Principia*, another attempt at a complete axiomatisation of mathematics was underway. In 1908 Zermelo published his system for the axiomatisation of set theory. In 1922, Fraenkel added to it and it is now known as Zermelo-Fraenkel (ZF) set theory. Adding the Axiom of Choice gives what is known as ZFC set theory. Zermelo thought that his axioms were tight enough to preclude contradictions while being broad enough to provide a foundation for 'normal' maths. They and others thought it was *complete* and *consistent* - but nobody could prove it.[29]

In response to the crisis of set theory and Russell's paradoxes, and in the spirit of the formalist *Zeitgeist*, Hilbert would develop and extend what became known as the Hilbert Programme, a programme for the creation of a sound epistemological foundation for all of mathematics by establishing a *complete*, *consistent*, and *decidable* set of axioms - and *decidable* meant a purely 'mechanical' procedure for deciding whether statements are true, 'a decision procedure'.

According to Zach, around 1917 Hilbert was hopeful that *Principia* had solved most of the problems but by 1920 realised it had not: 'the aim of reducing set theory, and with it the usual

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[28] In his *Autobiography*, Russell referred to meeting Einstein, Gödel and Pauli at the IAS in 1943: 'although all three of them were Jews and exiles and, in intention, cosmopolitans, I found that they all had a German bias for metaphysics [and] Gödel turned out to be an unadulterated Platonist.' When this was pointed out to Gödel in 1971, he drafted an unsent letter in which he wrote: 'I have to say (for the sake of truth)…, that I am not a Jew even though I don't think this question is of any importance… Concerning my "unadulterated" Platonism, it is no more "unadulterated" than Russell's own in 1921,' presumably a reference to Wittgenstein's *Tractatus*?

[29] What do we *now* think? ZFC is proved to evade the paradoxes of *naive set theory* revealed by Russell. It is thought to be consistent. However, it does not escape Gödel's *Incompleteness Theorems*. There will always be *undecidable* theorems in ZFC. The Continuum Hypothesis was shown by Gödel and Cohen to be an undecidable theorem in ZFC (on the assumption that ZFC *is* consistent). No 'normal' maths theorem, such as the Goldbach Conjecture, has been shown to be undecidable.

methods of analysis, to logic, has not been achieved today and maybe cannot be achieved at all' (Hilbert, 1920). Hilbert was also under attack from *Intuitionists* such as Brouwer[30]:

> '[Brouwer thought that] the human mind was strikingly limited in its ability to deal with infinite sets, where the logical principle of the excluded middle was, he argued, no longer applicable. When infinite sets of objects were under consideration, he argued that the usual dichotomy (a statement is either true or false) no longer applies. More precisely, he said that the only way to show that a set contains such-and-such an element was explicitly to construct such an element. It was not enough to prove that the assumption that the set does not contain this element leads to a contradiction.'[31]

Much to Hilbert's concern, Brouwer was supported by Hilbert's brightest student, Herman Weyl, who in 1920 had written a polemical paper, '*The new foundational crisis in mathematics*' which claimed that Cantor's infinite sets had brought mathematics to 'fog on fog'.[32]

> 'The antinomies of set theory are usually regarded as border skirmishes that concern only the remotest provinces of the mathematical empire and that can in no way imperil the inner solidity and security of the empire itself or of its genuine central areas. Almost all the explanations given by highly placed sources for these disturbances (with the intention of denying them or smoothing them over), however, lack the character of a clear, self-evident conviction, born of totally transparent evidence, but belong to that sort of half to three-quarters honest attempts at self-deception that one so frequently encounters in political and philosophical thought. Indeed, every earnest and honest reflection must lead to the realization that the troubles in the borderland of mathematics must be judged as symptoms, in which what lies hidden at the center of the superficially glittering and smooth activity comes to light - namely the inner instability of the foundations upon which the structure of the empire rests…
>
> 'We must again learn modesty. We wanted to storm the heavens and we have only piled fog on fog that cannot support anybody who tries in earnest to stand on them. What remains tenable could at first sight appear so meager that even the possibility of analysis is questionable; this pessimism is, however, unfounded, as the next section will show…'

Hilbert's star pupil also issued a rallying cry that upset his former teacher:

> 'For this order can not be maintained in itself, as I have now convinced myself, and Brouwer - that is the revolution!'

Hilbert replied in three lectures 1921-2 including his *Neubegrundung* (*New Grounding*). He set out a program for fully axiomatising mathematics based on a new 'proof theory' in which everything could be proved with a finite number of symbols.

> 'What Weyl and Brouwer do amounts in principle to following the erstwhile path of Kronecker: they seek to ground mathematics by throwing overboard all phenomena that make them uneasy and by establishing a dictatorship of prohibitions *à la* Kronecker. But this means to dismember and mutilate our science, and if we follow such reformers, we run the

[30] Brouwer's *fixed-point theorem* in topology would be used by von Neumann in the foundations of game theory, cf. an earlier blog in this series.

[31] Gray p.165 (2000)

[32] Weyl described the paper as 'a propaganda pamphlet' in a letter to Brouwer, 6 May 1920.

danger of losing a large number of our most valuable treasures... Weyl and Brouwer will be unable to push their programme through. No: Brouwer is not, as Weyl believes, the revolution, but only a repetition, with the old tools, of an attempted coup that, in its day, was undertaken with more dash, but nevertheless failed completely; and now that the power of the state has been armed and strengthened by Frege, Dedekind, and Cantor, this coup is doomed to fail...Weyl ... has artificially imported the vicious circle into analysis...

'I should like to regain for mathematics the old reputation for incontestable truth, which it appears to have lost as a result of the paradoxes of set theory...The method that I follow is none other than the axiomatic...The axiomatic method is and remains the indispensable tool, appropriate to our minds, for all exact research in any field whatsoever: it is logically incontestable and at the same time fruitful...

'Poincaré was from the start convinced of the impossibility of a proof of the consistency of the axioms of arithmetic. According to him, the principle of complete induction is a property of our mind - i.e. (in the language of Kronecker) it was created by God...

'... our question is influenced in its essence by the old attempts to ground number theory and analysis on set theory and set theory on pure logic.'

In this and other papers during the 1920's, Hilbert gave some version of this paragraph explaining his view of *finitstic reasoning.*

'[A]s a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication.'

In his 1925 *On the Infinite*, Hilbert explained why his Programme was of fundamental importance:

'Admittedly, the present state of affairs where we run up against paradoxes is intolerable. Just think, the definitions and deductive methods which everyone learns, teaches, and uses in mathematics, lead to absurdities!...

'Where else would reliability and truth be found if even mathematical thinking fails?... The definitive clarification of the *nature of the infinite* has become necessary, not merely for the special interests of the individual sciences, but rather for the *honour of human understanding* itself.'

In various papers and talks between 1925-1928 he described the search for a formal system expressing 'the whole thought content of mathematics in a uniform way' and described such a system as like 'a court of arbitration, a supreme tribunal to decide fundamental questions - on a concrete basis on which everyone can agree and where every statement can be controlled'.

Around 1924-1927, two colleagues of Hilbert, Ackerman and von Neumann, worked on a proof of the consistency of first-order logic. Ackerman thought he had a full proof in 1924 but realised a mistake limited his result. In 1925 (published 1927), von Neumann gave *a proof establishing the* *consistency of first-order logic* with the limitation that 'induction is applied only to quantifier-free formulas'. Ackerman subsequently refined his proof. By 1927, Hilbert thought that this work meant that there would soon be a final full proof of the consistency of of first-order logic. See Gödel's doctoral thesis below.

In his 1927 paper, *On the Foundations of Mathematics*, Hilbert described his developing 'proof theory' and addressed the Intuitionists:

> '[With 'proof theory'] I should like to eliminate once and for all the questions regarding the foundations of mathematics … by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science…
>
> 'The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds…'

He furiously rejected Brouwer's attack on the logical principle of the 'excluded middle'. It is unnecessary since 'it is not to be blamed in the least for the well-known paradoxes of set theory' which are, in fact, 'due merely to the introduction of inadmissible and meaningless notions which are automatically excluded from my proof theory'. Further, the principle is utterly vital to meaningful mathematics, its abandonment would be like 'proscribing the telescope to the astronomer or to the boxer the use of his fists' and would therefore 'be tantamount to relinquishing the science of mathematics altogether'. He described himself as 'astonished' that the idea had gained serious credibility and complained that 'the power of suggestion of a single man [Brouwer]' is creating 'most improbable and eccentric effects'.[33] He also predicted victory for his proof theory.

> 'Already at this time, I should like to assert what the final outcome will be: mathematics is a presuppositionless science. To found it I do not need God, as does Kronecker, or the assumption of a special faculty of our understanding attuned to the principle of mathematical induction, as does Poincaré or the primal intuition of Brouwer, or, finally, as do Russell and Whitehead, axioms of infinity, reducibility, or completeness, which in fact are actual, contentual assumptions that cannot be compensated for by consistency proofs.'[34]

In 1928, Hilbert made his Programme more precise by posing what became known as the *Entscheidungsproblem* – 'the decision problem'.

> 'The *Entscheidungsproblem* is solved when one knows a procedure by which one can decide in a finite number of operations whether a given logical expression is generally valid or is satisfiable. The solution of the *Entscheidungsproblem* is of fundamental importance for the theory of all fields, the theorems of which are at all capable of logical development from finitely many axioms.'[35]

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[33] As the battle between Hilbert and Brouwer raged, in 1928 Hilbert decided to purge Brouwer, who was a nationalist and unpopular, from the editorial board of the journal *Mathematische Annalen*, of which Hilbert was editor-in-chief.

[34] *On the Foundations of Mathematics* (1927).

[35] Church later wrote that the *Entscheidungsproblem* should be understood as a way 'to find an effective method by which, given any expression Q in the notation system, it can be determined whether or not Q is provable in the system.' Church solved this problem roughly the same time as Turing but in a very different way. Cf. below.

The *Entscheidungsproblem* is now often formulated in the modern terminology of computer science: *is there a general algorithm that can decide whether mathematical statements are valid?*

In a 1928 lecture in Bologna, Hilbert declared:

> 'With this new foundation of mathematics, which one can conveniently call proof theory, I believe the fundamental questions in mathematics are finally eliminated, by making every mathematical statement a concretely demonstrable and strictly derivable formula…
>
> '[I]n mathematics there is no *ignorabimus*, rather we are always able to answer meaningful questions; and it is established, as Aristotle perhaps anticipated, that our reason involves no mysterious arts of any kind: rather it proceeds according to formulable rules that are completely definite – and are as well the guarantee of the absolute objectivity of its judgement.'

Hilbert's approach is a reformulation of Leibniz's search for a *characteristica universalis, ars inveniendi* and *ars iudicandi* (see the previous paper in this series HERE). Von Neumann described the consequence of finding such a 'decision procedure' (1927):

> '… then mathematics, in today's sense, would cease to exist; its place would be taken by a completely mechanical rule, with the aid of which any man would be able to decide, of any given statement, whether the statement can be proven or not.'[36]

On 8 September 1930, Hilbert gave a farewell address in Königsberg and boldly stated:

> 'For the mathematician there is no *Ignoramibus*, and, in my opinion, not at all for natural science either... In an effort to give an example of an unsolvable problem, the philosopher Comte once said that science would never succeed in ascertaining the secret of the chemical composition of the bodies of the universe. A few years later this problem was solved…
>
> 'The true reason, according to my thinking, why Comte could not find an unsolvable problem is, in my opinion, that there is no unsolvable problem. In contrast to the foolish *Ignoramibus* ['*Ignoramus et ignoramibus*', We do not know and we shall never know], our credo avers: *Wir müssen wissen wir werden wissen* [We must know, We shall know].'[37]

Hilbert was confident that the combined attack of von Neumann, Bernays and Ackerman would very soon finally solve his Second Problem and the *Entscheidungsproblem*. He did not know that the day before an unknown Austrian, Kurt Gödel, had quietly announced his own momentous results to an initially underwhelmed conference.

Gödel and Hilbert would never meet or correspond. Bernays would later say that Hilbert was 'angry' when he learned of Gödel's results but he then ploughed on to explore new methods to save his Programme which did not work.

*

[36] This passage seems to be mistranslated in Mirowski p.79.

[37] A record of Hilbert saying this on German radio in 1930 still exists, on which one can hear him finish with '… *wir werden wissen*' and he laughs as the broadcast ends. This famous phrase is engraved on his tombstone.

Enter Kurt Gödel, 1930-31: incompleteness and undecidable problems

'Einstein had often told me that in the late years of his life he has continually sought Gödel's company in order to have discussions with him. Once he said to me that his own work no longer meant much, that he came to the Institute [IAS] merely um das Privileg zu haben, mit Gödel zu Fuss nach Hause gehen zu durfen [in order to have the privilege of walking home with Gödel].' Oskar Morgenstern (co-author with von Neumann of the first major work on Game Theory).

Gödel attended a conference in Königsberg, '*Epistemology of the Exact Sciences*', on 5-7 September 1930. On the 6th, Gödel gave a talk on his 1930 proof (for his doctoral dissertation[38]) of *the completeness of first order predicate calculus*.[39] On Sunday 7th September,[40] in a roundtable discussion including von Neumann and Carnap on the foundations of mathematics, he first announced his First Incompleteness Theorem. The transcript suggests that a comment he made elicited a response from von Neumann to which he replied something like:

> 'For of no formal system can one affirm with certainty that all contentual considerations are representable within it. Assuming the consistency of classical mathematics one can even give examples of propositions (and in fact of those of the type of Goldbach or Fermat) that, while contentually true, are unprovable in the formal system of classical mathematics…'

The transcript shows no further discussion of his remarks. None of the participants with the exception von Neumann seem to have realised the importance of what he was saying (one of the organisers did not mention the result in his report of the conference). Von Neummann immediately realised the significance of Gödel's bombshell, spoke to him at the end of the session, and on 20th November wrote to Gödel suggesting the Second Incompleteness Theorem. Three days earlier on 17th November, Gödel's paper *On Formally Undecidable Propositions of* Principia Mathematica *and Related Systems I* (1931),[41] had been received for publication; it was published in early 1931. Gödel replied to von Neumann that he had indeed proved it. Von Neumann realised that after years of effort, Hilbert's Programme was probably doomed.

The opening paragraphs explain Gödel's revolution:[42]

> 'The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia Mathematica* (PM) and the [Zermelo-Frankel] axiom system of set theory… These two systems are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to

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[38] His thesis was written in 1929 and was accepted by the university in February 1930 but he published a slightly different version in 1930 without the original introduction, so it is sometimes described as 1929 and sometime as 1930. He did not show his thesis to his supervisor Hahn until he had finished it and the latter's influence was merely to make some editorial suggestions.

[39] Franzen points out that there are many misunderstandings about this because of a natural misunderstanding of two different technical meanings of 'complete'. 'That [first order predicate calculus] is complete does not mean that some formal system is complete in the sense of Gödel's incompleteness theorem [1931].'

[40] This date is often mis-stated. I trust Feferman.

[41] A Part II was planned but not written.

[42] I use the translation by van Heijenhort which is preferred by scholars partly because it was carefully checked by Gödel himself. Unfortunately another translation is more widely used leading to some problems.

> decide *any* mathematical question that can at all be formally expressed in these systems. It will be shown below that this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integers that cannot be decided on the basis of the axioms. This situation is not in any way due to the special nature of the systems that have been set up but holds for a wide class of formal systems; among these … are all systems that result from [PM and ZF] through the addition of a finite number of axioms, provided no false propositions … become provable owing to the added axioms…'

What do *consistent, complete, and decidable* mean in Gödel's analysis?

> A formal system *S* is *consistent* if there is no A such that both A and ~A are theorems.
>
> A sentence is *undecidable* if neither A nor ~A is a theorem.
>
> *S* is *complete* if no sentence in the language of *S* is *undecidable* in S; otherwise it is *incomplete*. (Franzen)

Feferman, a leading scholar of Gödel, summarises his 1931 results thus:

> 'If S is a formal system such that:
> (i) the language of S contains the language of arithmetic,
> (ii) S includes PA [Peano Arithmetic], and
> (iii) S is consistent
> then there is an arithmetical sentence A which is true but not provable in S [first incompleteness theorem]… [and] the consistency of S is not provable in S [second incompleteness theorem].'

I will not go into the technical details of how he did it but this explanation from Yandell gives a hint. He mapped the formal system onto a list of natural numbers then used a similar method to Cantor's 1891 diagonal argument to show the existence of a true but unprovable statement. He established a system of numbering all statements in the formal language (a method that became known as 'Gödel numbering'), then created a list of the code numbers of provable statements, then used the diagonal method to find a statement that says something like 'the code number of the n^{th} statement on the diagonal is not on the master list of provable statements'; but this statement itself *is* the n^{th} statement, and therefore says something like 'I am not provable' (Yandell).

Gödel explained in a footnote the connection between his results and Cantor's set theory:

> 'The true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite, while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added...An analogous situation prevails for the axiom system of set theory.'

Despite Gödel showing that the *truth* of A could not be equated to the *provability* of A, it remained open whether nevertheless there was some general mechanical process to decide whether A or ~A or neither was formally provable in a given FAS. If Hilbert had been right and Gödel had proved the opposite of what he actually proved, then the *Entscheidungsproblem* definitely *would* be solvable. Given Gödel's proof its status was still unclear.

Long after Turing's 1936 proof (below) had extended the meaning of Gödel's results, Gödel described the meaning of his results in a Postscript added to his 1931 paper (28 August 1963), which states the two incompleteness theorems in plain language:

> 'In consequence of later advances, in particular of the fact that due to A.M. Turing's work a precise and unquestionably adequate definition of the general notion of formal system[43] can now be given, a completely general version of Theorems VI and XI [the two incompleteness theorems of the 1931 paper] is now possible. **That is, it can be proved rigorously that in *every* consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system**.'[44] (Bold added.)

*

Enter Turing, 1936: undecidable problems, uncomputable numbers, and Turing Machines

'He told me that the "main idea" of the paper came to him when he was lying in Grantchester meadows in the summer of 1935' (Gandy).

'[W]ith this concept [the Turing Machine] one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion.' Gödel.

Turing's *On Computable Numbers, With an Application to the Entscheidungsproblem* (1936) combined Gödel's work with the thought experiment of a particular type of 'computer', which is now known as a Turing Machine (TM), to mechanise the analysis of the *Entscheidungsproblem*.[45] Turing demonstrated the equivalence of the *Entscheidungsproblem* and the question: is there a general programme (algorithm) which can decide in advance whether any particular programme halts (terminates) or not ('the Halting Problem'). He showed that a 'Universal Turing Machine' (UTM) can simulate the action of any individual Turing Machine and it can *compute anything that can be computed using the rigorous apparatus of a suitable formal axiomatic system*. However, this does *not* mean that a UTM can solve any mathematical problem. Some problems are absolutely unsolvable by a TM or a human using certain types of formal axiomatic system. They are known as '*uncomputable*'.

The thought experiment he created for his proof became known as a Turing Machine (TM). A TM consists of:

- (i) an infinite tape (its infinite memory) broken into cells which bears a language (of finite alphabet), say a binary code, into the machine;
- (ii) a read / write head that reads / writes the finite alphabet symbol by symbol, and which exists in some finite number of states;
- (iii) a set of transition rules from one state to another (programme).

One of the leading computer scientists, Bennett, describes the concept:

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[43] Footnote in original: 'In my opinion the term 'formal system' or 'formalism' should never be used for anything but this notion [ie. Turing's]…' The 'characteristic property' of formal systems 'is that reasoning in them, in principle, can be completely replaced by mechanical devices.'

[44] *Collected Works*, Volume I, p. 195.

[45] 'He told me that the 'main idea' of the paper came to him when he was lying in Grantchester meadows in the summer of 1935' (Gandy).

'[C]omputation is a stepwise manipulation of symbols following fixed rules without the intervention of judgement or chance. A Turing Machine consists of an infinite *tape*, on which a finite set of symbols (typically 0,1, and blank) can be read and written by a *head*, which itself is capable of existing in only some finite number of internal states. At each step, the machine's action is determined by the currently scanned tape symbol and current head state, and consists of halting forthwith, or of writing a new tape symbol, shifting one space left or right along the tape, and changing to a new head state. This cycle of operations is repeated *ad infinitum*, or until the machine halts.'

Some TMs with a few dozen head states are *universal* 'able to perform any computation and simulate any computer, even one much larger and more complicated than itself' – they are 'Universal Turing Machines' (UTMs).

'A [UTM] works by keeping on its tape a coded representation of the head state, head position, and tape contents of the larger machine being simulated, and breaks each step of the larger machine into many steps simple enough to be executed by the [UTM].'[46]

Turing applied these TMs to what is now known as the 'Halting Problem' (Turing used different terminology). The Halting Problem is easy to understand in principle. We know some computations halt: e.g. what is the first even number after 10? Answer = 12. We know some computations do not halt: e.g. find an odd number that is the sum of two even numbers. Some computations are a mystery. For example, Fermat's Last Theorem can be presented as: find $x^n + y^n = z^n$ for $n>2$. For centuries we did not know whether this would halt or not. Now that Andrew Wiles recently solved it we know that this computation will never halt.

Turing demonstrated that the *Entscheidungsproblem* was equivalent to the question: *is there a general programme (algorithm) which can decide in advance whether any particular programme halts?* The answer is No, there is no TM that can solve the Halting Problem and therefore the *Entscheidungsproblem* has no solution.

Turing's paper partly rests on Cantor's work on the non-denumerability of the reals. Like Gödel, he uses Cantor's diagonal method in his proof that 'computable' numbers do not include all definable numbers and are therefore, unlike the reals, denumerable and this means that a computer cannot solve the *Entscheidungsproblem*. Although Turing's terminology is slightly different, it is important to note that a 'computer' can mean *both* a human computer with pencil and paper and what we think of as a computer. He showed that each can 'compute' the same category of numbers: the 'computable' numbers for a human are the same category as the formal definition he gives for a TM.

'A number is computable if its decimal can be written down by a machine…

'I show that certain large classes of numbers are computable. They include for instance the real parts of all algebraic numbers, … the numbers π, e, etc [NB. π and e are transcendental, so the computable numbers include some, but not all, transcendentals]. The computable numbers do not, however include all definable numbers…

[46] 'Besides being able to simulate other [TMs], [UTMs] have been found to be able to simulate a great many other models of computation, including ones with multidimensional tape, random-access memory, sequential arrays of logic gates, and various kinds of parallel processing.' Bennett

'Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless denumerable. [NB. The set of all real numbers *is* non-denumerable as Cantor proved.]…'

Turing first defined rigorously a class of functions (the computable functions) that corresponded to the functions that a human could compute. This is often now referred to as *the Church-Turing Thesis: a TM can perform any calculation that any human computer can do*, or 'the mathematical class of "computable functions" corresponds *exactly* to the class of functions that we would naturally regard as computable' (Nielsen).[47] In his PhD thesis three years later, Turing defined the idea as: 'We shall use the expression 'computable function' to mean a function calculable by a machine.' *Turing then showed that no function in the class of computable functions could solve the Entscheidungsproblem.*

Von Neumann summarised Turing's conclusion for the non-mathematician:

'The problem is to determine, for a class of logical expressions or propositions, whether there is a mechanical method for deciding whether an expression of this class is true or false… Turing proved that there is something for which you cannot construct an automaton; namely, you cannot construct an automaton which can predict in how many steps another automaton, which can solve a certain problem, can actually solve it… *In other words, you can build an organ which can do anything that can be done, but you cannot build an organ which can tell you whether it can be done… This is connected with … the results of Gödel.* The feature is just this, that you can perform within the logical type that's involved everything that's feasible, but the question of whether something is feasible within a type belongs to a higher logical type.'[48]

When the editor of von Neumann's notes reviewed them after his death, he wrote to Gödel asking about the references to his work. Gödel replied:

'I have some conjecture as to what von Neumann may have had in mind… [A] complete epistemological description of a language *A* cannot be given in the same language *A*, because the concept of truth of sentences of *A* cannot be defined in *A*. It is this theorem that is the true reason for the existence of undecidable propositions in the formal systems containing arithmetic.'

One of the standard graduate textbooks on computer theory summarises Turing's proof in terms of software verification:.

'[You have] a computer program and a precise specification of what that program is supposed to do (e.g. sort a list of numbers). You need to verify that the program performs as specified… Because both the program and the specification are mathematically precise objects, you hope to automate the process of verification by feeding these objects into a suitably programmed computer. However, you will be disappointed. The general problem of software verification is not solvable by computer.' (Sipser, p.181.)

[47] Church proved the same thing as Turing at almost the same time but in a very different way that is formally equivalent but did not have the same attraction for most mathematicians or the same obvious implications for practical computers, as Church himself accepted. Gödel developed the concept of 'recursive functions' which is also equivalent to the Church and Turing concepts. Cf. Copeland p. 41ff and Deutsch's 1985 below. Other models of computation that are equivalent to Turing's are known as 'Turing complete'.

In terms of the basic concepts of number described in the first section, we can summarise some of our knowledge after Turing's paper:

1. The set of all computer programs is *denumerable*.
2. The set of all *computable reals* is *denumerable*.
3. The set of all *uncomputable reals* is *non-denumerable*.
4. Most *reals* are *uncomputable*, infinitely more than are computable.
5. Every non-computable number is *transcendental*.[49]

Although Turing invented these concepts to solve Hilbert's problem, it was obvious to some that they provided a logical structure for building real computers. After publishing his most famous paper, he went to Princeton where he discussed computation with von Neumann and wrote another fascinating paper for his PhD thesis, *'Systems of Logic Based on Ordinals' (1938)*. Von Neumann offered him a job but Turing preferred to return to Britain where he soon embarked on his heroic work at Bletchley Park.

Turing and von Neumann turned Turing's abstract demonstration of the possibilities of programming computers into engineering reality. After the war, von Neumann succeeded in persuading both the IAS and different branches of the US government to invest in developing computers, partly based on his argument about their usefulness in simulating atomic explosions and therefore their vital role viz Stalin. Unfortunately the British government botched the development of computers and did not build on Turing's legacy. (See next blog in this series.)

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David Deutsch extends the Gödel-Turing results to the quantum realm

In 1985, the physicist David Deutsch published *'Quantum theory, the Church-Turing principle and the universal quantum computer'*. This was soon after Feynman's 1981 paper, *Simulating Physics with Computers*, which first explored the possibility of *quantum computation*. Deutsch extended the Church-Turing thesis into a physical principle: *the idea (unproved but very plausible) that every physical process can be simulated by a UTM.*

1. The Church-Turing thesis is that every function which would naturally be regarded as computable can be computed by the UTM.
2. 'The conventional, non-physical view of [the thesis] interprets it as the quasi-mathematical conjecture that all possible formalizations of the intuitive mathematical notion of 'algorithm' or 'computation' are equivalent to each other. But we shall see that it can also be regarded as asserting a new physical principle, which I shall call the Church-Turing principle to distinguish it from other implications and connotations…'
3. The thesis is 'very vague by comparison with physical principles such as the laws of thermodynamics or the gravitational equivalence principle. But it will be seen below that my statement of the Church-Turing principle is manifestly physical, and unambiguous. I shall show that it has the same epistemological status as other physical principles.'
4. The Principle can be stated as: 'Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means.' This approach is 'both better defined and more physical than Turing's own way of expressing it' and avoids terminology like 'naturally be regarded'.

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[49] Another consequence of the Gödel-Turing results was that it pointed a way to an answer to Hilbert's Tenth Problem - is there a process by which it can be determined in a finite number of operations whether a Diophantine equation is solvable in rational integers? By 1970 this was shown to be unsolvable (see here).

5. 'Classical physics and the universal Turing machine, because the former is continuous and the latter discrete, do not obey the principle, at least in the strong form above.'
6. 'A class of model computing machines that is the quantum generalization of the class of Turing machines is described, and it is shown that quantum theory and the "universal quantum computer" are compatible with the principle.'
7. 'Computing machines resembling the universal quantum computer could, in principle, be built and would have many remarkable properties not reproducible by any Turing machine. These do not include the computation of non-recursive functions, but they do include "quantum parallelism", a method by which certain probabilistic tasks can be performed faster by a universal quantum computer than by any classical restriction of it.'
8. Finally, Deutsch discusses the implications of quantum computation for the long-standing problem, fought over since Einstein and Bohr, of interpreting what quantum mechanics is really telling us about 'reality': 'The intuitive explanation of these properties places an intolerable strain on all interpretations of quantum theory other than Everett's [the Many Worlds interpretation].'
9. The quantum generalisation of what Deutsch called the Church-Turing Principle is now known as the Church-Turing-Deutsch Principle.
10. These ideas are also bound up with ideas concerning entropy, information, and complexity, including the work of Bennett, Landauer *et al* on resolving the problem of Maxwell's Demon and the 'reversibility' of computation with quantum mechanics.[50] 'To view the Church-Turing hypothesis as a physical principle does not merely make computer science a branch of physics. It also makes part of experimental physics into a branch of computer science.'

Rewording the Principle helps realise how shocking it is: *there is a single physical system which can be used to simulate any other system in the Universe*.

The man who wrote *the* textbook on quantum computers, Michael Nielsen, wrote a great non-technical essay on Deutsch's work.

> 'Turing needed to show that his universal computer could perform any conceivable algorithmic process. This wasn't easy. Until Turing's time, the notion of an algorithm was informal, not something with a rigorous, mathematical definition. Mathematicians had, of course, previously discovered many specific algorithms for tasks such as addition, multiplication and determining whether a number is prime. It was pretty straightforward for Turing to show that those known algorithms could be performed on his universal computer. But that wasn't enough.
>
> 'Turing also needed to convincingly argue that his universal computer could compute any algorithm whatsoever, including all algorithms that might be discovered in the future. To do this, Turing developed several lines of thought, each giving an informal justification for the idea that his machine could compute any algorithmic process. Yet he was ultimately uncomfortable with the informal nature of his arguments, saying "All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically…"

Deutsch 'made the observation that algorithmic processes are necessarily carried out by physical systems. These processes can occur in many different ways: A human being using an abacus to multiply two numbers is obviously profoundly different from a silicon chip running a flight simulator. But both are examples of physical systems, and as such they are governed by the same underlying

[50] Bennett explained the story of this extremely advanced subject in a great piece: *Demons, Engines and the Second Law* (Scientific American, 1987).

laws of physics. With this in mind, Deutsch stated the following principle …: "Every finitely realizable physical system can be perfectly simulated by a universal model computing machine operating by finite means." In other words, take any physical process at all, and you should be able to simulate it using a universal computer… In a sense, if you had a complete understanding of the machine, you'd understand all physical processes.'

'Deutsch's principle goes well beyond Turing's earlier informal arguments. If the principle is true, then it automatically follows that the universal computer can simulate any algorithmic process, since algorithmic processes are ultimately physical processes. You can use the universal computer to simulate addition on an abacus, run a flight simulator on a silicon chip, or do anything else you choose.

'Furthermore, unlike Turing's informal arguments, Deutsch's principle is amenable to proof. In particular, we can imagine using the laws of physics to deduce the truth of the principle. That would ground Turing's informal arguments in the laws of physics and provide a firmer basis for our ideas of what an algorithm is.

'In attempting this, it helps to modify Deutsch's principle in two ways. First, we must expand our notion of a computer to include quantum computers. This doesn't change the class of physical processes that can be simulated in principle, but it does allow us to quickly and efficiently simulate quantum processes. This matters because quantum processes are often so slow to simulate on conventional computers that they may as well be impossible. Second, we must relax Deutsch's principle so that instead of requiring perfect simulation, we allow simulation to an arbitrary degree of approximation. That's a weaker idea of what it means to simulate a system, but it is likely necessary for the principle to hold.

'With these two modifications, Deutsch's principle becomes:

Every finitely realizable physical system can be simulated efficiently and to an arbitrary degree of approximation by a universal model (quantum) computing machine operating by finite means.

'No one has yet managed to deduce this form of Deutsch's principle from the laws of physics. Part of the reason is that we don't yet know what the laws of physics are…

'While Preskill and his collaborators haven't yet succeeded in explaining how to simulate the full Standard Model, they have overcome many technical obstacles to doing so. It's plausible that a proof of Deutsch's principle for the Standard Model will be found in the next few years…

'We can imagine using computers to simulate not only our own laws of physics, but maybe even alternate physical realities. In the words of the computer scientist Alan Kay: "In natural science, Nature has given us a world and we're just to discover its laws. In computers, we can stuff laws into it and create a world." Deutsch's principle provides a bridge uniting the sciences of the natural and the artificial. It's exciting that we're nearing proof of this fundamental scientific principle.'

Elsewhere, Nielsen explained how finding a process that cannot be simulated on a computer, which is conceivable, would re-open aspects of this debate.

> 'Turing's arguments were a creditable foundation for the Church-Turing thesis, but they do not rule out the possibility of finding somewhere in Nature a process that cannot be simulated on a Turing machine. If such a process were to be found, it could be used as the basis for a computing device that could compute functions not normally regarded as computable. This would force a revision of the Church-Turing thesis, although perhaps not a

scrapping of the thesis entirely – what is more likely is that we would revise the definition of the Church-Turing thesis.'

I will return to this subject in a later note on quantum computers and P=NP?

Illustration: the execution on a quantum computer of the following computer program ***is*** *a performance of the famous Einstein-Podolski-Rosen experiment (Deutsch)*

```
begin
  int n = 8 * random;        % random integer from 0 to 7 %
  bool x, y;                 % bools are 2-state memory elements %
  x := y := false;           % an irreversible preparation %
  V(8, y);                   % see equation (2.15) %
  x eorab y;                 % perfect measurement (2.14) %
  if V(n, y) ≠               % measure y in random direction %
    V(n, x)                  % and x in the parallel direction %
    then print((”Quantum theory refuted.”))
    else print((”Quantum theory corroborated.”))
  fi
end
```

*

The Gödel-Turing results, physics, and AI

It has been claimed by many that the Gödel-Turing results somehow prove that certain types of physical theories are impossible. This has even been claimed by prominent scientists such as Freeman Dyson and Stephen Hawking. However, the experts on Gödel-Turing say that such opinions are usually a misunderstanding.

For example, Freeman Dyson wrote the following:

> 'Now I claim that because of Gödel's theorem, physics is inexhaustible too. The laws of physics are a finite set of rules, and include the rules for doing mathematics, so that Gödel's theorem applies to them. The theorem implies that even within the domain of the basic equations of physics, our knowledge will always be incomplete.'[51]

Saul Feferman, chief editor of Gödel's *Collected Works*, replied pointing out that while it was true that no future physical theory could escape Gödel's incompleteness theorems, this is not at all the same thing as prohibiting a supposedly 'complete' set of physical laws:[52]

> '… if the laws of physics are formulated in an axiomatic system *S* which includes the notions and axioms of arithmetic as well as physical notions such as time, space, mass, charge, velocity, etc., and if *S* is consistent, then there are propositions of higher arithmetic which are undecidable by S. But this tells us nothing about the specifically physical laws encapsulated in S, which could conceivably be complete as such.'

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[51] NY Review of Books, 13 May 2004. Feferman reply, 15 July 2004

[52] NYRB weblink above.

Feferman made the further point that all the maths required by physics requires only the ZFC axiom system used for all normal maths 'and there is not the least shred of evidence' that a stronger system is needed: 'In fact, it has long been recognized that much weaker systems than that suffice for scientific applications.'

Feferman then distinguished between a) the untroubled use of ZFC by applied maths and b) the metamathematical problems of pure maths and set theory.

> 'It is entirely another matter whether, and in what sense, pure mathematics needs new axioms beyond those of the Zermelo-Fraenkel system; that has been a matter of some controversy among logicians.'

Dyson replied to the NYRB:

> 'I am grateful to Solomon Feferman for explaining why we do not need Gödel's theorem to convince us that science is inexhaustible.'

Hawking in a 2002 speech, 'Gödel and the End of Physics'[53], made a similar argument: 'Maybe it is not possible to formulate the theory of the universe in a finite number of statements. This is very reminiscent of Gödel's theorem.' Hawking argued that 'if there are mathematical results that cannot be proved, there are physical problems that cannot be predicted' and gives an example of trying to solve the Goldbach Conjecture in physical form by arranging blocks of wood.

As Franzen stresses, Gödel's and Turing's 1931/1936 only tell us about formal systems with arithmetic.

> 'The basic equations of physics … cannot decide every arithmetical statement [and could not and will not however they develop], but whether or not they are complete considered as a description of the physical world, and what completeness might mean in such a case, is not something that the incompleteness theorem tells us anything about…
>
> 'Our predictions of the outcome of physical experiments using arithmetic are based on the premise that arithmetic provides a good model for the behaviour of certain actual physical systems with regard to certain observable properties… The relevant description of the physical world amounts to the assumption that this premise is correct. The role of the arithmetical statements is as a premise in the application of this description to arrive at conclusions about physical systems…
>
> '… [N]othing in the incompleteness theorem excludes the possibility of our producing a complete theory of stars, ghosts and cats, all rolled into one, as long as what we say about stars, ghosts and cats can't be interpreted as statements about the natural numbers.'[54]

It is also claimed (most famously by Roger Penrose) that the Gödel-Turing results 'prove' that intelligent machines are impossible. Most of the leading scholars of the subject do not agree with such claims (cf. Franzen) and Gödel and Turing rejected such arguments.

Despite the fact that 'the incompleteness theorem shows that we cannot formally specify the sum total of our mathematical knowledge' (Franzen), and proves the existence of 'undecidable

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[53] http://www.damtp.cam.ac.uk/strings02/dirac/hawking/

[54] Franzen p.88-90.

propositions' and 'uncomputable numbers', it should be stressed that so far these things have been confined to the far edge of mathematical logic - no 'normal' mathematical problem has been proved to be 'undecidable'.

> '… so far, no unsolved problem of prior mathematical interest like these [Goldbach Conjecture, Riemann Hypothesis] has even been shown to be independent of Peano Arithmetic. The true statement shown to be unprovable by Gödel is just contrived to do the job; it doesn't have mathematical interest on its own.' Feferman
>
> 'No arithmetical conjecture or problem that has occurred to mathematicians in a mathematical context, that is, outside the special field of logic and the foundations or philosophy of mathematics, has ever been proved to be undecidable in ZFC...
>
> '… a proof that the twin prime conjecture is undecidable in ZFC would be a mathematical sensation comparable to the discovery of an advanced underground civiliazation on the planet Mars.' (Franzen)[55]

It is therefore vital to remember a point rarely made when one comes across general statements about the Gödel-Turing results: they apply to a limited and very carefully defined area of mathematics and many statements that you read about them are incorrect extensions to fields beyond their proper domain.

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Conclusion

One can see why von Neumann called Gödel's results 'a land mark which will remain visible far in space and time'. To have a definitive proof answering such a profound question concerning the nature of knowledge was astonishing and it is no surprise the result seeped into the broader culture, albeit usually mangled. In future blogs, I will relate some of the history concerning the practical development of computers for problems concerning the prediction of complex systems.

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Bibliography (least specialist to most)

The Story of Mathematics, Marcus du Sautoy (2008). Non-specialist, suitable for school children.

Number, Dantzig (updated version 2007). Einstein called it 'beyond doubt the most interesting book on the evolution of mathematics which has ever fallen into my hands.' It is deeper than du Sautoy and requires more effort but a lot can be understood without specialist education.

Gödel's Theorem: An Incomplete Guide to its Use and Abuse, Franzen. The best book about Gödel's Theorem (according to the editor of Gödel's Collected Works) which explains why almost everything one reads about it is wrong.

The essential Turing, Copeland. Reprints Turing's papers with expert commentary.

[55] In 2013 the twin prime conjecture was finally proved. A relatively obscure mathematician had been working on it in secret for a decade and announced his amazing result. A Polymath project swiftly improved it.

What is mathematics?, Courant (updated version 1996). 'It is a work of high perfection.' Weyl. This is a specialist book though some of it is accessible to a curious and determined non-specialist.

The Collected Works of Kurt Gödel (4 volumes), edited by Feferman. Exceptional scholarship for the expert. I like flicking through it even though there is little I can understand.